THE COUNTING PRINCIPLE

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TARGET AUDIENCE: Further Education and Training educators
DURATION: 1 hour
MAXIMUM NUMBER OF PARTICIPANTS: 30

MOTIVATION FOR THIS WORKSHOP

Probability is a relatively new topic in School mathematics and was also not part of a considerable number of teachers' pre- service training and studies. Teaching Data handling and Probability, poses a real challenge to some educators because their knowledge in this content area is limited. In this workshop we will focus on The Counting Principle, a topic in the grade 12 curriculum. The following key concepts are covered in this topic (as indicated in the Curriculum Statement).

- *Probability problems using Venn diagrams, tree diagrams, two way contingency tables and other techniques (like the fundamental counting principle) to solve probability problems (where events are not necessarily independent).*
- *Apply the fundamental counting principle to solve probability problems*

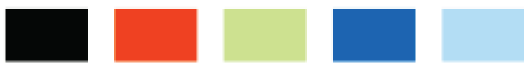
COUNTING PRINCIPLES

Introductory example: Suppose you are buying a new car.

There are **2** body styles:

sedan or hatchback

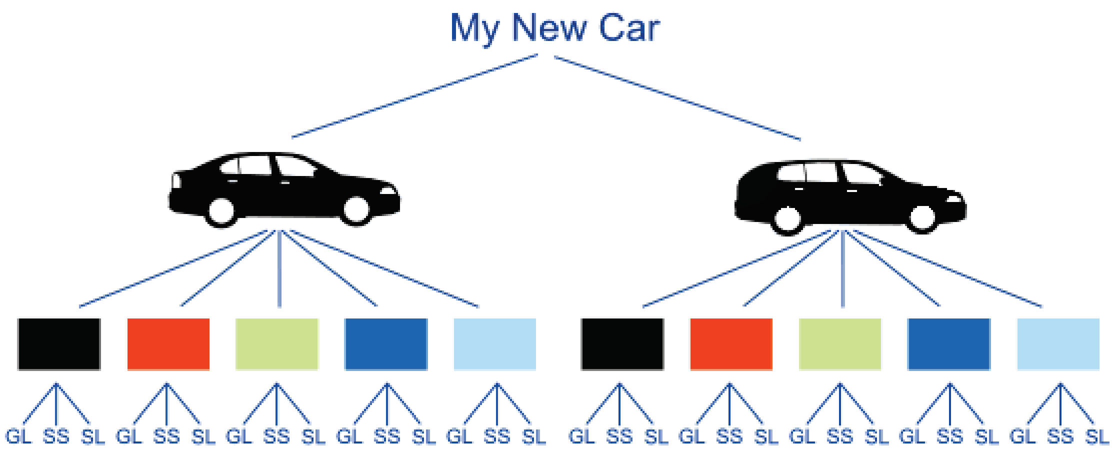
There are **5** colours available:



There are **3** models:

- GL (standard model),
- SS (sports model with bigger engine)
- SL (luxury model with leather seats)

How many total choices?



It is possible to list all possible outcomes using a tree diagram. When you have many possible outcomes, a tree diagram can become very messy and it becomes difficult to count the possible outcomes.

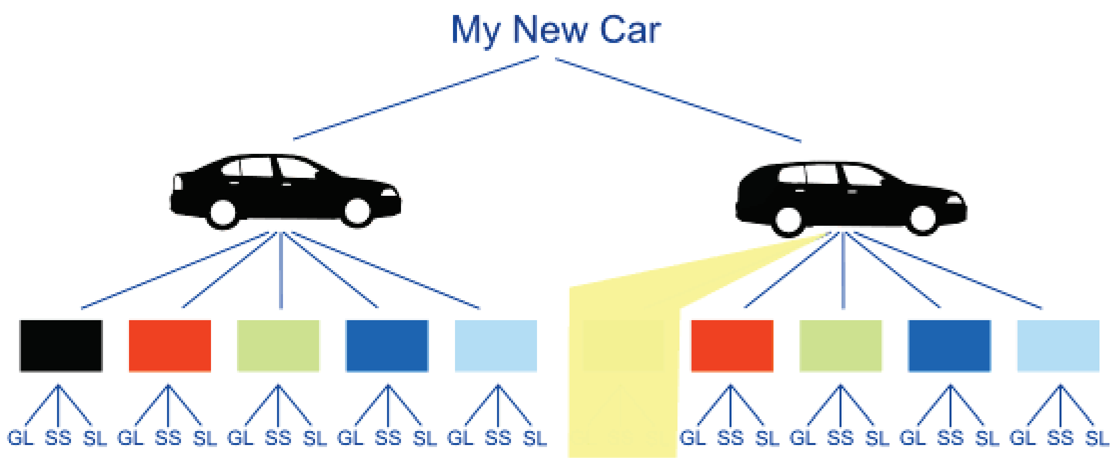
Counting principles help you to count the possible outcomes without drawing a tree diagram.

Independent or Dependent?

If one choice affects another choice (i.e. **depends** on another choice), then a simple multiplication is not right.

You are buying a new car ... but ...

The salesman says **"You can't choose black for the hatchback"** ... well then things change!



You now have only 27 choices.

Because your choices are **not independent** of each other.

But you can still make your life easier with this calculation:

$$\text{Choices} = 5\times3 + 4\times3 = 15 + 12 = 27$$

Arrangements with repeats

It will assist in understanding the counting principles by thinking back to the tree diagrams that you used in previous grades.

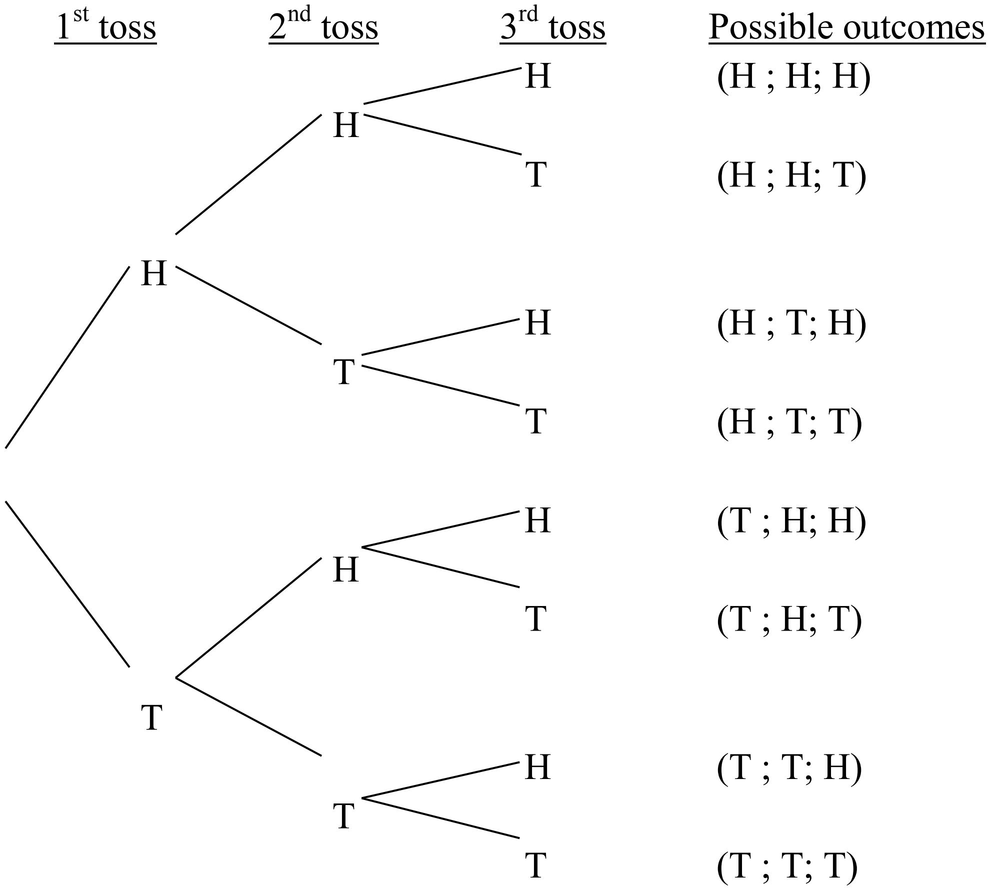
Example 1

a) A coin is tossed twice. How many outcomes are there?
b) A coin is tossed three times. How many outcomes are there?
c) A coin is tossed four times. How many outcomes are there?
d) A coin is tossed twenty times. How many outcomes are there?

Solution

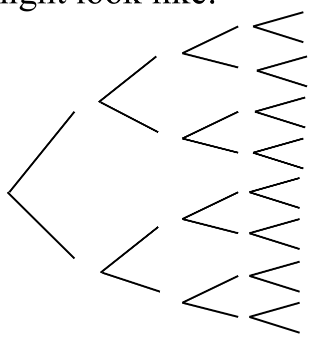
a)

b)



There are $2 \times 2 \times 2 = 8$ outcomes in total.

c) It becomes more difficult to draw a tree diagram to show all the possible outcomes when a coin is tossed 4 times. Below is a simple sketch of what one might look like:



When a coin is tossed four times:
There are two possible outcomes for the first toss.
There are two possible outcomes for the second toss.
There are two possible outcomes for the third toss.
There are two possible outcomes for the fourth toss.
There are $2 \times 2 \times 2 \times 2 = 16$ outcomes in total.

d) It is not easy to draw a tree diagram for 20 tosses, but you should be seeing a pattern.
If a coin is tossed 20 times, there will be $2 \times 2 \times 2$…. (to 20 terms)
Number of possible outcomes = ……………………………….

- The ***fundamental counting principle*** is a quick method for calculating numbers of outcomes using multiplication.

- The fundamental counting principle states:

  Suppose there are n_1 ways to make a choice, and for each of these there are n_2 ways to make a second choice, and for each of these there are n_3 ways to make a third choice, and so on.
  The product $n_1 \times n_2 \times n_3 \times \ldots \times n_k$ is the number of possible outcomes.

In simple language the fundamental counting principle says:
"If you have several stages of an event, each with a different number of outcomes, then you can find the TOTAL number of outcomes by multiplying the number of outcomes of each stage."

Arrangements without Repeats

Sometimes we have examples where an event can only be used once.

Activity 1

a) Two counters marked A and B are randomly drawn from a box. When a counter is taken, it is not returned. How many ways can these letters be drawn, i.e. how many possible outcomes are there?

b) Three counters marked A, B and C are randomly drawn from a box. When a counter is taken, it is not returned. How many possible outcomes are there?

Factorial Notation

- The arrangement of numbers $4 \times 3 \times 2 \times 1$ can be written as 4!
  - You say '***4 factorial***'.
  - $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$
  - $n! = n \times (n-1) \times (n-2) \times (n-3) \times (n-4) \times \ldots\ldots \times 4 \times 3 \times 2 \times 1$
- Factorial notation is used for finding the total number of outcomes without repeats.
- Most scientific calculators have a factorial key.

On the Casio fx-82ZA PLUS, the factorial key ($x!$) is next to the (x^{-1}).
To calculate 6!, enter the number (e.g. 6) then press [SHIFT] [x^{-1}] ($x!$) [=]

On the Sharp EL-W535HT, the factorial key ($n!$) is next to the 1.
To calculate 6!, enter the number (e.g. 6) then press: [2nd F] [4] ($n!$) [=]

Activity 2

a) A three digit code is made up of numbers 3, 5 and 7. The digits may be repeated. How many different codes are possible?

b) A three digit code is made up of numbers 3, 5 and 7. Each digit is used only once. How many different codes are possible?

- WHEN REPEATS ***ARE ALLOWED***, *you find the total number of outcomes by multiplying the number of possible outcomes in each stage of an event.*

- WHEN REPEATS ***ARE NOT ALLOWED***, *you find the total number of outcomes by multiplying the number of possible outcomes* ***that are left*** *in each stage of an event.*

- Sometimes you don't want to arrange all the items, but only some of them.

Activity 3

1) Write each of these as a product of digits (e.g. $3! = 3 \times 2 \times 1$)
   a) 4!
   b) 7!

2) Use your calculator to determine each of the following (e.g. 5! = 120)
   a) 6!
   b) 8!

3) The following menu is offered at a restaurant:

Starter	Main	Dessert
Chicken livers Tomato soup Salad	Peri-peri chicken Lamb chops Beef rump Fish and chips	Fruit salad and ice-cream Chocolate pudding

   If one starter, one main and one dessert is selected from the menu. How many combinations of starter, main and dessert could be chosen?

4) Given the numbers: 0; 1; 2; 3; 4 and 5
   a) How many 4 digit numbers can be formed if the first digit may not be 0 and the numbers may not be repeated?
   b) How many of these numbers will be divisible by 5?

5) The soccer coach needs the goal posts to be moved on the field. He randomly chooses five boys out of a group of twenty to help him. How many different groups of 5 boys can be selected?

Special Conditions

Sometimes when we are counting the number of arrangements, we are given special conditions, for example similar groups must be arranged together, or two or more elements must be put together in the arrangement.

Activity 4

A photograph needs to be taken of the Representative Council of Learners (RCL) at a school. There are three girls and two boys in the RCL and all of them need to sit in one row for the photograph.

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a) Suppose there is no restriction on the order in which the RCL sits. In how many ways can the RCL be arranged in a row?
b) Suppose the President and the Vice-President of the RCL must be seated next to each other, in how many different ways can the RCL be arranged in a row?
c) Suppose all the girls must sit next to each other, and all the boys must sit next to each other. In how many different ways can the RCL be arranged in a row?

Identical Items in a list

Consider how many arrangements of the letters there are in the word LEEK. Here is a list of some of the possible arrangements:

LEEK	LEKE	LEEK	LEKE	
LKEE	LKEE	ELEK	EELK	Etc

- Because the letter E is repeated, we *cannot* say that there are 4! different arrangements. In fact, because 2 letters are repeated, there is half the number of different arrangements than there would be if all four letters were different.

We say that

- If there are n different items that are all different, then there are $n \times (n-1) \times (n-2)$ ... n terms or $n!$ arrangements.

- If there are n different items, but ***one item is repeated twice***, then there are
  $n \times (n - 1) \times (n - 2)$ ... n terms **divided by** 2 or 2! arrangements.
- If there are n different items, but ***one item is repeated three times***, then there are $n \times (n - 1) \times (n - 2)$ ... n terms **divided by** $3 \times 2 \times 1$ or 3! arrangements.

Activity 5

a) In how many ways can you arrange the letters in the word MEDIAN?..
b) In how many ways can you arrange the letters in the word DATA? …………………………………………………
c) In how many ways can you arrange letters in the word PERCENTILE? ……………………………………
d) In how many ways can you arrange the letters in the term CENSUS@SCHOOL? ……………………………………

Using Counting Principles to Find Probability

- You can use these counting principles to find the number of possible outcomes, and you can also use them to find the number of favourable outcomes.

- When you know the number of possible outcomes and the number of favourable outcomes, you can work out the probability of the favourable event using the formula

Probability of a favourable event =

$$= \frac{\text{Number of favourable outcomes}}{\text{Number of possible outcomes}}$$

Activity 6

1) Suppose a four-digit number is formed by randomly selecting four digits without repetition from 1; 2; 3; 4; 5; 6; 7 and 8.

   What is the probability that the number formed lies between 4 000 and 5 000?

2) What is the probability that a random arrangement of the letters in the name 'PHILLIPINE', start and end in 'L'?

REFERENCES

Statistics South Africa (2013) (unpublished). Data Handling & Probability, Grades 10 -12

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Website: www.mathsisfun.com