

Why Does the 'Tip-and-Times' Rule Work?

Faaiz Gierdien

Research Unit for Mathematics Education, University of Stellenbosch (RUMEUS)

faaiz@sun.ac.za

This article is about understanding why the 'tip-and-times' or 'invert-and-multiply' rule for dividing by a fraction works. I will illustrate the reasoning behind this rule using proper and improper fractions as well as a practical example using ribbon material. The reasoning entails using multiplying or dividing by 1 and equivalence (Askew, 2008, 2013) or equivalent ways to rewrite fractions or whole numbers. In the case of the practical example using ribbon material, a central idea is changing or switching the measuring unit. The two examples used come from primary school pre-service teachers (PSTs) whom I teach.

Let's begin by considering the following improper and proper fractions that have whole numbers as numerators and denominators: $\frac{7}{4}$, $\frac{5}{4}$, $\frac{4}{4}$, $\frac{3}{4}$ and $\frac{1}{4}$. The 'bar' in each fraction implies a quotient, meaning that we divide the numerator by the denominator. This division is the same as following the procedure: invert ('tip') the denominator and multiply ('times') it by the numerator. In each case the denominator (4) is 'tipped' and becomes $\frac{1}{4}$. Table 1 illustrates the 'tip-and-times' procedure for dividing two whole numbers.

<i>Improper/proper fractions</i>	<i>Dividing the whole numbers</i>	<i>'Tip-and-times' equivalent</i>
$\frac{7}{4} = 1\frac{3}{4}$	$7 \div 4$	$7 \times \frac{1}{4}$
$\frac{5}{4} = 1\frac{1}{4}$	$5 \div 4$	$5 \times \frac{1}{4}$
$\frac{4}{4} (= 1)$	$4 \div 4$	$4 \times \frac{1}{4}$
$\frac{3}{4}$	$3 \div 4$	$3 \times \frac{1}{4}$
$\frac{1}{4}$	$1 \div 4$	$1 \times \frac{1}{4}$

TABLE 1: 'Tip-and-times' rule when dividing by a whole number.

What about instances where the numerators are whole numbers but the denominators are fractions as in $2 \div \frac{1}{5}$, $2 \div \frac{2}{5}$, $2 \div \frac{3}{5}$ and $2 \div \frac{4}{5}$? The choice of keeping the 2 the same and varying the fraction is simply to make the exercise systematic so as to notice any patterns in the answers.

<i>Numerator whole number; denominator fraction</i>	<i>Applying equivalence to the whole number numerators</i>		<i>Using 'tip-and-times' rule to get the same answers</i>
$\frac{2}{\frac{1}{5}} = 2 \div \frac{1}{5}$	$\frac{10}{5} \div \frac{1}{5}$	$\frac{10 \div 1}{5 \div 5} = \frac{10 \div 1}{1}$	$\frac{10}{5} \times \frac{5}{1} = \frac{10}{1}$
$\frac{2}{\frac{2}{5}} = 2 \div \frac{2}{5}$	$\frac{10}{5} \div \frac{2}{5}$	$\frac{10 \div 2}{5 \div 5} = \frac{10 \div 2}{1}$	$\frac{10}{5} \times \frac{5}{2} = \frac{10}{2}$
$\frac{2}{\frac{3}{5}} = 2 \div \frac{3}{5}$	$\frac{10}{5} \div \frac{3}{5}$	$\frac{10 \div 3}{5 \div 5} = \frac{10 \div 3}{1}$	$\frac{10}{5} \times \frac{5}{3} = \frac{10}{3}$
$\frac{2}{\frac{4}{5}} = 2 \div \frac{4}{5}$

TABLE 2: 'Tip-and-times' rule when dividing a whole number by a fraction.

In the second column from the left we see how 2 is written as the equivalent fraction $\frac{10}{5}$. The choice of $\frac{10}{5}$ is based on the denominator of the fraction by which it is being divided. In the third column, numerator is divided by numerator while denominator is divided by denominator, the latter in each case resulting in $5 \div 5$, i.e. 1. The last column shows how the tip and times procedure results in the same answers. Readers are encouraged to complete the table and to make their own list of examples where the numerator is a whole number and the denominator a fraction.

What about instances where the numerators and denominators are both fractions, as in $\frac{3}{4} \div \frac{1}{3}$, $\frac{3}{4} \div \frac{2}{3}$, $\frac{3}{4} \div \frac{4}{3}$ and $\frac{3}{4} \div \frac{5}{3}$? The choice of dividing by different fraction values of thirds is based on making the exercise systematic.

<i>Numerator & denominator as fractions</i>	<i>Applying equivalence to the numerator & denominator</i>		<i>Using 'tip-and-times' rule to get the same answers</i>
$\frac{3}{4} \div \frac{1}{3}$	$\frac{3}{4} \div \frac{1}{3} = \frac{9}{12} \div \frac{4}{12}$	$\frac{9 \div 4}{12 \div 12} = \frac{9 \div 4}{1}$	$\frac{3}{4} \times \frac{3}{1} = \frac{9}{4}$
$\frac{3}{4} \div \frac{2}{3}$	$\frac{3}{4} \div \frac{2}{3} = \frac{9}{12} \div \frac{8}{12}$	$\frac{9 \div 8}{12 \div 12} = \frac{9 \div 8}{1}$	$\frac{3}{4} \times \frac{3}{2} = \frac{9}{8}$
$\frac{3}{4} \div \frac{4}{3}$

TABLE 3: 'Tip-and-times' rule when dividing a fraction by another fraction.

Equivalence is used in the second column from the left to re-write each of the two fractions so that they have a common denominator. In the third column, numerator is divided by numerator while denominator is divided by denominator, the latter in each case resulting in $12 \div 12$, i.e. 1. Dividing by 1 means that one only has to consider what happens in the numerator. In the last column on the left we notice that we get

the same answers when we use the ‘tip-and-times’ approach. Dividing by and multiplying by 1 is therefore important in understanding how and why the ‘tip-and-times’ rule works.

We now examine an instance where the ‘tip-and-times’ rule is explained in a slightly different way. Figure 1 shows a pre-service teacher’s procedure and reasoning behind using the ‘tip-and-times’ procedure to determine $\frac{3}{8} \div \frac{4}{6}$.

Examples (the deeper meaning [skipped steps left out])

$$\frac{3}{8} \div \frac{4}{6}$$

$$= \frac{3 \times 6}{8 \times 6} \div \frac{4 \times 8}{6 \times 8}$$

$$= \frac{3 \times 6}{8 \times 6} \div \frac{(4 \times 8)}{(6 \times 8)}$$

$$= \frac{3 \times 6}{4 \times 8} \div \frac{(4 \times 8)}{(6 \times 8)}$$

— the divide line is the same as the \div sign

$$= \frac{3}{8} \times \frac{6}{4} = \frac{18}{32} = \frac{9}{16}$$

The ‘tip and times’ rule works because you use the identity element for multiplication, this is where you multiply by 1, which gives equivalence. So element for multiplication was used where you have $\frac{6}{6}$ and $\frac{8}{8}$. The first line is $\div 4$ the last line is $\times \frac{6}{4}$, which is the \times then tip therefore it will work for any division sum with fractions, just leave out the in between steps, because you do the same for every sum.

$$\frac{2}{7} \div \frac{3}{9} = \frac{2}{7} \times \frac{9}{3} = \frac{18}{21} = \frac{6}{7} \quad \text{! it will always work!}$$

FIGURE 1: A PST’s procedure and reasoning behind the ‘tip-and-times’ rule.

This particular pre-service teacher makes use of the identity element for multiplication, *viz.* 1, which she writes as $\frac{6}{6}$ and $\frac{8}{8}$ in the second line of her process. Through a particular way of grouping the numbers she gets her denominator to be 1, *viz.*, $8 \times 6 \div (6 \times 8)$ as in her fourth line: $\frac{3 \times 6 \div (4 \times 8)}{1}$. From her procedure

we see $\frac{3 \times 6 \div (4 \times 8)}{1}$ as being equivalent to $\frac{3 \times 6}{4 \times 8}$. Using equivalence, as in the fact that 4×8 gives the same answer as 8×4 in the denominator, she rewrites $\frac{3 \times 6}{4 \times 8}$ as $\frac{3}{8} \times \frac{6}{4}$. She concludes with a seeming sense of excitement that her procedure “will always work!”, giving her own example: $\frac{2}{7} \div \frac{3}{9} = \frac{2}{7} \times \frac{9}{3}$. She ends

by recommending “just leave out the in between steps,” i.e. simply apply the ‘tip-and-times’ rule or procedure. Readers are encouraged to make up their own examples and follow a similar process to that shown in Figure 1 to explore why the ‘tip-and-times’ rule works.

Let us now move to a practical example using ribbon material from which bows are to be cut. The question is: If you have 3 metres of ribbon, and each bow needs $\frac{4}{5}$ metres of ribbon, how many bows can you make? Figure 2 shows a pre-service teacher's approach to answering the question.

(d) If you have 3m of ribbon and it take $\frac{4}{5}$ to make a ribbon bow, how many bows can you make?

visually:

1m

1m

e 1m

Therefore you can make 3 ribbons. $\frac{3}{4}$ is left over.

numerically:

$$\begin{aligned}
 & 3 \div \frac{4}{5} \\
 & = \frac{15}{5} \div \frac{4}{5} \\
 & = \frac{15}{5} \times \frac{5}{4} \\
 & = \frac{75}{20} \\
 & = 3 \frac{15}{20} \\
 & = 3 \frac{3}{4}
 \end{aligned}$$

FIGURE 2: A PST's procedure and reasoning for measuring bows from ribbon.

The PST showed the 3 metres of ribbon “visually” as three separate 1 metre strips. Each of the strips she divides into fifths and then proceeds to ‘measure off’ units of four fifths of a metre. At this point her measuring unit changes from fifths to fourths, which she indicates in her first strip of ribbon. She marks off these three sections as 1, 2 and 3. The remaining ribbon amounts to a measurement in of fourths – i.e. three quarters of a full ribbon. She thus arrives at a final answer of 3 full ribbons and a partial ribbon of three quarters the full length. With respect to the ‘tip-and-times’ procedure, the ‘tip’ occurs when the measuring unit shifts from fifths to fourths. The ‘tip’ is thus about moving from ribbons to bows and vice versa. One bow has a length of $\frac{4}{5}$ of a metre whereas one metre of ribbon amounts to $\frac{5}{4}$, i.e. $1\frac{1}{4}$, of a bow. Table 4 captures this ‘tip’.

<i>Ribbon</i>	<i>Bow</i>
$\frac{4}{5}$ metre of ribbon	1 bow
1 metre of ribbon	$\frac{5}{4} = 1\frac{1}{4}$ bows

TABLE 4: Relationship between ribbon and bow.

Figure 2 shows this particular PST's procedure to arrive at her answer of $3\frac{3}{4}$. Her numerical process is based on equivalent fractions, specifically $3 = \frac{15}{5}$, $1 = \frac{4}{4}$ and $1 = \frac{5}{5}$. The process she employs in Figure 2 is a somewhat extended version of the process illustrated in Table 3, *viz.*:

$$\begin{aligned}
 3 \div \frac{4}{5} &= \frac{15}{5} \div \frac{4}{5} \\
 &= \frac{15 \div 4}{5 \div 5} \\
 &= \frac{15 \div 4}{1} \\
 &= \frac{15}{4} \\
 &= 3\frac{3}{4}
 \end{aligned}$$

To illustrate the 'tip-and-times' procedure we can re-write the second-last line $\frac{15}{4}$ as $\frac{3 \times 5}{4}$ which is equivalent to $3 \times \frac{5}{4}$. Readers are encouraged to answer the following questions, using both visual and numerical means:

- If you have 3 metres of ribbon, how many bows of length $\frac{2}{5}$ metres can you make?
- If you have 3 metres of ribbon, how many bows of length $\frac{3}{5}$ metres can you make?
- If you have 3 metres of ribbon, how many bows of length $\frac{6}{5}$ metres can you make?

In conclusion, the 'tip-and-times' rule works whenever we divide, whether or not we have a whole number in the denominator. The exception is of course when the denominator is zero. The rule is based on using equivalence, equivalent fractions and the fact that multiplying or dividing a number or magnitude by 1 leaves the value unchanged.

REFERENCES

- Askew, M. (2008). Mathematical discipline knowledge requirements for prospective primary mathematics teachers, and the structure and teaching approaches of programs designed to develop that knowledge. In P. Sullivan & T. Wood (Eds.), *International handbook of mathematics teacher education: Vol. 1: Knowledge and beliefs in mathematics teaching and teaching development* (pp. 13–35). Rotterdam: Sense Publishers.
- Askew, M. (2013). Big Ideas in primary mathematics: issues and directions, *Perspectives in Education*, 31(3), 5–18.