Investigating Factors

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SUMMING THE RECIPROCALS OF THE FACTORS OF A NUMBER

In 2007 the following question appeared in the Third Round of the Junior Section of the South African Mathematics Olympiad:

> "The sum of the factors of 120 is 360. Find the sum of the reciprocals of the factors of 120."

Given $1 + 2 + 3 + \cdots + 40 + 60 + 120 = 360$ we are thus required to find:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{40} + \frac{1}{60} + \frac{1}{120}$$

Now, since each of the factors of 120 will divide into 120, the LCD of the above sum is 120. We can thus calculate the sum as follows:

$$Sum = \frac{120 + 60 + 40 + \cdots + 3 + 2 + 1}{120} = \frac{360}{120} = 3$$

Note that the sum of the reciprocals is the original sum (360) divided by the original number (120). Let's investigate this scenario more generally.

OBSERVATIONS REGARDING SQUARE AND NON-SQUARE NUMBERS

Let $1\ ;\ a_1\ ;\ a_2\ ; \ldots ;\ a_{m-1}\ ;\ a_m\ ;\ n$ be the factors of a number n, increasing in size from 1 to n, and let their sum equal x. Then we have:

$$1 + a_1 + a_2 + \cdots + a_{m-1} + a_m + n = x$$

The sum of the reciprocals will be:

$$\frac{1}{1} + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{m-1}} + \frac{1}{a_m} + \frac{1}{n}$$

Now, since $1\ ;\ a_1\ ;\ a_2\ ; \ldots ;\ a_{m-1}\ ;\ a_m\ ;\ n$ are all the factors of n in increasing order, pairing off and multiplying the smallest with the largest, the second smallest with the second largest etc., we have:

$$n = 1 \times n = a_1 \times a_m = a_2 \times a_{m-1} = a_3 \times a_{m-2} \text{ etc.}$$

If the number of factors is even, then:

$$\text{Sum} = \frac{n}{1n} + \frac{a_m}{a_1 a_m} + \frac{a_{m-1}}{a_2 a_{m-1}} + \cdots + \frac{a_2}{a_{m-1} a_2} + \frac{a_1}{a_m a_1} + \frac{1}{n}$$

$$= \frac{n + a_m + a_{m-1} + \cdots + a_2 + a_1 + 1}{n}$$

$$= \frac{x}{n}$$

What would happen if there were an odd number of factors? After pairing off the factors, smallest with largest etc. as previously done, then it's clear that the middle factor will be left unpaired. Let this middle factor be a_i. The sum of the reciprocals will thus be:

$$\frac{1}{1}+\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_i}+\cdots+\frac{1}{a_{m-1}}+\frac{1}{a_m}+\frac{1}{n}$$

We thus have, for an odd number of factors:

$$\begin{aligned}\text{Sum} &= \frac{n}{1n}+\frac{a_m}{a_1a_m}+\frac{a_{m-1}}{a_2a_{m-1}}+\cdots+\frac{a_i}{a_ia_i}+\cdots+\frac{a_2}{a_{m-1}a_2}+\frac{a_1}{a_ma_1}+\frac{1}{n}\\ &= \frac{n+a_m+a_{m-1}+\cdots+a_i+\cdots+a_2+a_1+1}{n}\\ &= \frac{x}{n}\end{aligned}$$

This proves that in general if the sum of the factors of a given number, n, equals x , then the sum of the reciprocals of the factors of n is given by the quotient $\frac{x}{n}$.

From the above it should also be clear that $n = {a_i}^2$ showing that if a number has an *odd* number of factors then it must be a perfect square.

FURTHER OBSERVATIONS

In LTM No. 3 of 2006 there appears an article titled "HCF, LCM and the number of factors" (Fresen, Fresen & Heidema, 2006). In this article the authors proved, using the fact that a natural number can always be expressed as a product of its prime factors, that for a number x expressed as a product of its prime factors, that is $x = {p_1}^{n_1} \times {p_2}^{n_2} \times \ldots \times {p_k}^{n_k}$, the total number of factors of x is given by:

$$(n_1+1)(n_2+1)\ldots(n_k+1)$$

Now, given that $x = {p_1}^{n_1} \times {p_2}^{n_2} \times \ldots \times {p_k}^{n_k}$, in order for x to be a perfect square, *all* of $n_1, n_2, \ldots, n_k$ must be *even*, thus $(n_1+1)(n_2+1)\ldots(n_k+1)$ must be $(odd)(odd)\ldots(odd)$ confirming that a perfect square must have an odd number of factors. Conversely, if x is not a perfect square then *one or more* of $n_1, n_2, \ldots, n_k$ must be *odd*, making *one or more* of (n_1+1) ; (n_2+1) ; … ; (n_k+1) *even*, thus making the product even and thereby confirming that a number which is not a perfect square has an even number of factors.

By way of example, consider 36 which is a perfect square. 36 has 9 factors (1, 2, 3, 4, 6, 9, 12, 18 and 36) and can be prime factorised as $2^2 \times 3^2$. Using the formula $(n_1+1)(n_2+1)\ldots(n_k+1)$ to calculate the number of factors gives $(2+1)(2+1) = 9$, which as expected is odd. Now consider 120 which is not a perfect square. 120 has 16 factors (1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60 and 120) and can be prime factorised as $2^3 \times 3^1 \times 5^1$. The number of factors is $(3+1)(1+1)(1+1) = 16$, which is indeed even.

REFERENCES

Fresen, D., Fresen, J., & Heidema, J. (2006). HCF, LCM and the number of factors. *Learning and Teaching Mathematics, 3*, 16-17.