Two Vignettes from a Grade 8 Classroom

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INTRODUCTION

It has been a few years since I last taught a Grade 8 Maths class. This year I am fortunate to be teaching one of two parallel top sets in Grade 8, and I have so enjoyed the energy and level of engagement of the class. There is hardly a lesson that goes by without something interesting cropping up – whether it be a thought-provoking mathematical observation, a slightly different technique or approach to a question, or something completely tangential to what we're doing in class. These moments are incredibly special, and I try to capitalise on them as much as possible. In this short article I share two such moments that arose serendipitously while we were working on a module about fractions and ratio.

VIGNETTE 1

The module on fractions that we were busy with in class included proper and improper fractions; mixed numbers; adding, subtracting, multiplying and dividing fractions; squaring, cubing and rooting fractions; decimal fractions and percentages. For most of the class the majority of this content was revision of what had previously been covered at primary school, so we were moving quite swiftly through the material. The class was busy with an exercise on percentages when one pupil put up his hand and said "*Sir, I remember being told that percentages are reversible, is that true?*" I wasn't entirely sure what he meant, so I asked him to explain with an example. One of the questions he had recently done was to calculate 60% of 80, so he used that as an example. "*If I work out 60% of 80, will that be the same as 80% of 60?*". What a great question! Using one of the techniques we had discussed in class he calculated both as follows:

$$\begin{aligned}60\% \text{ of } 80 &= \frac{60}{100} \times \frac{80}{1} \\ &= \frac{6}{10} \times \frac{80}{1} \\ &= \frac{6}{1} \times \frac{8}{1} \\ &= 48\end{aligned} \qquad\qquad \begin{aligned}80\% \text{ of } 60 &= \frac{80}{100} \times \frac{60}{1} \\ &= \frac{8}{10} \times \frac{60}{1} \\ &= \frac{8}{1} \times \frac{6}{1} \\ &= 48\end{aligned}$$

Sure enough, the answers were indeed the same. Trying one or two more examples showed that it worked for other numbers as well. By this stage more and more pupils were getting involved, and the dialogue became one of – will this *always* work? And if so, *why*? One pupil conjectured that the case of 1% might be a problem, but others quickly established that this would still work, e.g. 1% of 80 is 0,8 and 80% of 1 is also 0,8. What about 0%? Well, 0% of 80 is 0 and 80% of 0 is also 0. So far so good. "*What about negative percentages?*" one pupil asked. We chatted a bit about what a negative percentage really means, and why we generally don't make use of them (e.g. it's easier to talk about a decrease of 5% rather than saying -5%), but of course it would nonetheless still work with negative percentages.

In terms of trying to make sense of why A% of B will always equal B% of A, I drew pupils' attention to the calculation of 60% of 80 and 80% of 60 as shown on the previous page. Pupils quickly realised that since both the 60 and the 80 were in the numerator, and that order doesn't matter with multiplication, it was clear from the first line that the two answers would definitely be the same. This was the real "*Aha!*" moment, and it was a small step from that to illustrating algebraically why it would *always* work:

$$a\% \text{ of } b = \frac{a}{100} \times \frac{b}{1} = \frac{b}{100} \times \frac{a}{1} = b\% \text{ of } a$$

What became particularly exciting is that pupils then started using this generality for simplifying the working in some of the exercises. For example, in a question that asked them to calculate 8% of 200, they realised that it would be much simpler just to calculate 200% of 8, since all they would need to do is double 8 to get the answer of 16 directly. Brilliant! Instead of calculating 6% of 150 they could simply work out 150% of 6 by multiplying it by 1½ to get 9. Not only did this method allow them to simplify a few of the question to something that they could work out mentally without having to write down any working details, but it also got them to engage with the questions on a deeper cognitive level as they had to think about when it would be helpful to restructure the question as opposed to leaving it in its original form. This heightened level of engagement was wonderful to see, as it added value as well as an unexpected depth to the exercise.

VIGNETTE 2

After the module on fractions we moved on to a section involving ratio and proportion. One of the first exercises was a revision activity where pupils had to express ratios in their simplest form. Some of the questions included ratios of fractions and mixed numbers that had to be appropriately manipulated into an equivalent ratio involving integer values. By way of example, one such question was to write $\frac{3}{4} : \frac{5}{6}$ as a ratio in simplest form. The approach followed was first to write the two fractions in an equivalent form with a common denominator, and then to scale each fraction by multiplying through by this common denominator.

$$\frac{3}{4} : \frac{5}{6} \;\rightarrow\; \frac{9}{12} : \frac{10}{12} \;\rightarrow\; 9:10$$

Having just finished the section on fractions, pupils were very comfortable with this approach and started working rapidly through the set of exercises. One question that involved mixed numbers was to write the ratio $2\frac{4}{5} : 4\frac{2}{3}$ in simplest form. The approach was to write the mixed numbers as improper fractions, then to write the improper fractions as equivalent fractions with a common denominator, and then finally to scale them up to arrive at an equivalent ratio expressed only with integers. One pupil was carrying out this process as follows:

$$2\frac{4}{5} : 4\frac{2}{3} \;\rightarrow\; \frac{14}{5} : \frac{14}{3} \;\rightarrow\; \dots$$

At this point he noticed that the two improper fractions had the same numerator, so he stopped working and called me over. His question: "*Sir, if the two fractions have the same numerator, will the ratio just be the ratio of the denominators?*" Another great question! I asked him to think about the relative size of the two original mixed numbers, and he quickly realised that they couldn't be in the ratio 5:3 since the first mixed number was clearly smaller than the second. He was a bit disappointed by this as he thought he had found something really interesting!

I stayed by his side and asked him to carry on with the question:

$$2\frac{4}{5} : 4\frac{2}{3} \;\rightarrow\; \frac{14}{5} : \frac{14}{3} \;\rightarrow\; \frac{42}{15} : \frac{70}{15} \;\rightarrow\; 42{:}\,70$$

I then asked him if he couldn't take it one step further and simplify the ratio 42:70. Realising that both numbers were divisible by 7 he quickly finished off the simplification process:

$$2\frac{4}{5} : 4\frac{2}{3} \;\rightarrow\; \frac{14}{5} : \frac{14}{3} \;\rightarrow\; \frac{42}{15} : \frac{70}{15} \;\rightarrow\; 42{:}\,70 \;\rightarrow\; 6{:}\,10 \;\rightarrow\; 3{:}\,5$$

When he spotted that the final ratio was simply the reversed ratio of the two denominators of the improper fractions he quite literally jumped out of his chair! What a wonderful moment of discovery!

The question now became – will this *always* work, and if so *why*? I quickly gave him another ratio to consider, namely $\frac{7}{2} : \frac{7}{3}$. Since the two numerators were the same, his conjecture was that the simplified ratio would simply be 3:2, i.e. the reversed ratio of the two denominators. We quickly confirmed that this was indeed the case:

$$\frac{7}{2} : \frac{7}{3} \;\rightarrow\; \frac{21}{6} : \frac{14}{6} \;\rightarrow\; 21{:}\,14 \;\rightarrow\; 3{:}\,2$$

We were then able to unpack why this works, and to prove it in general using an algebraic approach. We first talked about ratios as being equivalent to fractions – for example, a ratio of 5:6 can be thought of as the fraction $\frac{5}{6}$. When we have a ratio of two fractions, we can think of this ratio as the first fraction divided by the second, i.e. $\frac{7}{2} : \frac{7}{3}$ can be thought of as $\frac{7}{2} \div \frac{7}{3}$. We can then 'invert and multiply' to get $\frac{7}{2} \times \frac{3}{7}$ which simplifies to $\frac{3}{2}$, i.e. the ratio 3:2. From this example, and the 'invert and multiply' process, it became clear why the second denominator becomes the first integer of the final ratio. Following the numeric example he was able to make sense of a general scenario involving the ratio of two fractions with the same numerator:

$$\frac{a}{b} : \frac{a}{c} \;\rightarrow\; \frac{a}{b} \div \frac{a}{c} \;\rightarrow\; \frac{a}{b} \times \frac{c}{a} \;\rightarrow\; \frac{c}{b} \;\rightarrow\; c{:}\,b$$

Thus, the ratio of two fractions with the same numerator can be simplified directly to the ratio of the two denominators in the reverse order. What a beautiful piece of mathematical 'discovery'.

CONCLUDING COMMENTS

What I particularly like about these two classroom episodes is that they were completely unplanned and entirely serendipitous. The first arose from a chance comment from a pupil that could easily have gone unexplored and unrecognised, and the second arose from an observation of a particular question that just happened to have a very specific structure. It is often these unexpected moments where some of the best learning takes place in the classroom, and one needs constantly to have one's ears out on stalks listening for such chance comments and observations so that one can capitalise on them. I am so often in awe of just how much curiosity and mental acuity young pupils have. Tapping into this incredible potential is what the teaching game is all about.