Is a Trigonometric Proof Possible for the Theorem of Pythagoras?

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INTRODUCTION

In the previous century, Elisha Loomis (1968, p. 244), in his famous book of over 250 proofs of the theorem of Pythagoras, reasoned that no trigonometric proof of the Pythagorean theorem was possible. He argued as follows:

> "There are no trigonometric proofs [of the Pythagorean theorem], because all of the fundamental formulae of trigonometry are themselves based upon the truth of the Pythagorean theorem; because of this theorem we say $\sin^2 A + \cos^2 A = 1$, etc. Trigonometry *is* because the Pythagorean theorem *is*."

However, is it really true that no trigonometric proof is possible? Some older AMESA members may recall that in the late 1980s, Dexter Luthuli and I had a vigorous discussion in the Letters to the Editor of the journal *Pythagoras* precisely about this issue. While Dexter steadfastly maintained the viewpoint of Loomis (1968), I argued that a trigonometric proof was possible as long as one did not commit a circular argument by using the Pythagorean trigonometric identity ($\sin^2 A + \cos^2 A = 1$).

Basically, my argument was that any similarity proof of Pythagoras could be rewritten in terms of the basic, introductory definitions of the sine, cosine and tangent ratios for right-angled triangles. Below are more details about my argument as well as an example showing how a familiar similarity proof of Pythagoras can be seen to be equivalent to one in terms of the basic trigonometric ratios.

A SIMILARITY PROOF TRANSFORMED INTO A TRIGONOMETRIC ONE

First of all, it is important to note, as also pointed out in De Villiers (2022), that similarity forms the fundamental basis of trigonometry.

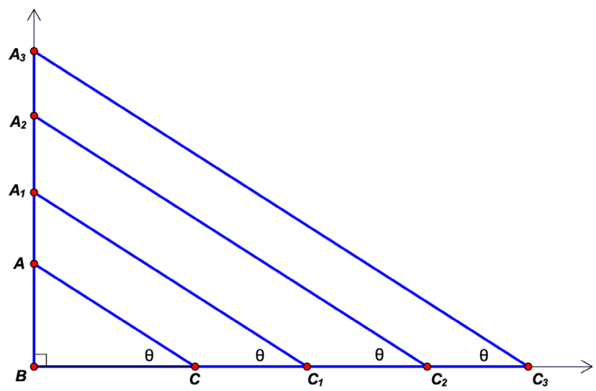

FIGURE 1

For example, as shown in Figure 1, all right triangles A_nBC_n with $\angle B = 90°$ and a given $\angle C_n = \theta$, are similar, since two corresponding angles are equal. Hence, the ratios of the sides, AB/BC, AB/AC and BC/AC of all right triangles for a given angle θ are constant, and this gives us the three basic trigonometric ratios, namely, tangent, sine and cosine. To paraphrase the quote from Loomis earlier, we could therefore say: "Trigonometry *is* because similarity *is*."

Let us now consider a well-known similarity proof of the theorem of Pythagoras. Although it's not known exactly what proof Pythagoras himself gave, according to historian van der Waerden (1978), this type of proof might have been what he could have produced.

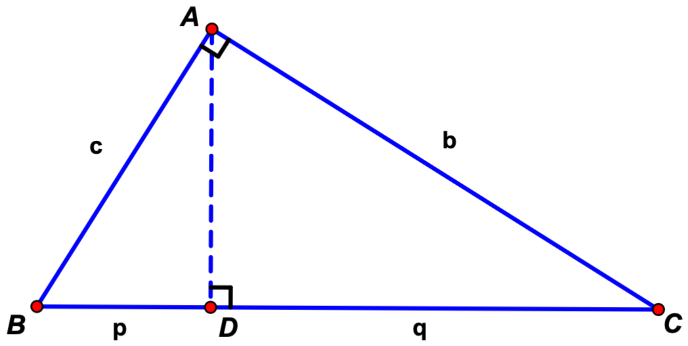

FIGURE 2

Proof:
Consider a right-angled triangle ABC with $\angle A = 90°$ as shown above in Figure 2. Drop a perpendicular from A to intersect BC at D, and let $AB = c$, $AC = b$, $BD = p$, $DC = q$ and $BC = a$. I now present the similarity proof in the left column of the table below, whereas the equivalent trigonometric version is shown in the right column.

$\Delta ABD\ ///\ \Delta CBA$ (2 corresponding $\angle$s equal)	In ΔABD, $\cos B = \frac{BD}{AB}$; In ΔCBA, $\cos B = \frac{AB}{CB}$
$\Rightarrow \frac{BD}{AB} = \frac{AB}{CB} \Rightarrow c^2 = p \times a$	$\Rightarrow \frac{BD}{AB} = \frac{AB}{CB} \Rightarrow c^2 = p \times a$
$\Delta ACD\ ///\ \Delta BCA$ (2 corresponding $\angle$s equal)	In ΔACD, $\cos C = \frac{DC}{AC}$; In ΔBCA, $\cos C = \frac{AC}{BC}$
$\Rightarrow \frac{DC}{AC} = \frac{AC}{BC} \Rightarrow b^2 = q \times a$	$\Rightarrow \frac{DC}{AC} = \frac{AC}{BC} \Rightarrow b^2 = q \times a$
$\Rightarrow b^2 + c^2 = q \times a + p \times a$ $= (q + p) \times a$ $= a^2$ Q.E.D.	$\Rightarrow b^2 + c^2 = q \times a + p \times a$ $= (q + p) \times a$ $= a^2$ Q.E.D.

Clearly the trigonometric proof[1] in the column on the right is merely a different notational rendition of the similarity proof on the left. And since it does not at all involve the identity $\sin^2 A + \cos^2 A = 1$, no circular argument occurs. It is therefore equivalent to, and as valid, as the corresponding similarity proof.

Some may argue that the proof in the column on the right is merely a similarity proof 'in disguise'. Personally, however, I would prefer to call it a 'trigonometric proof' since it explicitly uses the definition of the cosine ratio in terms of a right triangle. And basically, any valid proof of Pythagoras using similarity can be translated in the same way into an equivalent trigonometric proof, and readers are invited to explore converting some examples of their own.

[1] At Wikipedia at https://en.wikipedia.org/wiki/Pythagorean_theorem it is stated that basically the same trigonometric proof as this one was apparently also given by Albert Einstein.

TRIGONOMETRIC PROOF BY ZIMBA & LUZIA

Zimba (2009) provided an ingenious proof of the trigonometric identity $\sin^2 x + \cos^2 x = 1$ independently of the Pythagorean theorem by applying the compound angle formula for the cosine function, namely $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. He then used the independently proven identity $\sin^2 x + \cos^2 x = 1$ to prove the Pythagorean theorem. Along similar lines, Luzia (2015) gives other trigonometric proofs by using the half angle formula. However, it is important to note that both authors carefully restricted the domain of all angles involved to positive acute angles[2], and defined the sine and cosine functions as ratios of the sides of right-angled triangles. As such, the proofs are therefore based on similarity as before.

A RECENT TRIGONOMETRIC PROOF

Earlier this year two high school students, Calcea Johnson and Ne'Kiya Jackson, from St. Mary's Academy in New Orleans, caused quite a media stir by their discovery of a trigonometric proof of the Pythagorean theorem. The two students were even invited to the Spring Southeastern Sectional Meeting of the American Mathematical Society to present their results. For more information, and for reconstructions of their proof, readers are invited to view the *YouTube* videos by MathTrain (2023) and polymathematic (2023) – references and URLs are provided at the end of the article.

While the students' proof appears to be quite original and creative, it also uses similarity and the basic right triangle definitions of trigonometric ratios exactly as in the example above (as well as the sine rule and the sum of an infinite geometric series). Roughly, the idea of their proof is as follows.

Rough outline of proof by Johnson & Jackson

Consider Figure 3. The students start out with a right-angled triangle ABC with sides a, b, c and angles α and β as indicated. The students then reflected the right triangle in side b and drew a perpendicular to BB' at B'. The extension of the reflected side c was next drawn to meet with the perpendicular to AB constructed at B to form a large right triangle ABD[3]. The students then created an infinite series of converging similar right triangles as shown.

By applying the sine rule to triangle ABB' with respect to vertices A and B, then simplifying by substituting the value of $\sin\beta$ in right triangle ABC, they arrived at the following equation:

$$\sin(2\alpha) = \frac{2a.\sin\beta}{c} = \frac{2ab}{c^2} \ \dots (1)$$

Next the students determined the lengths of BD and AD by calculating the respective sums of the converging geometric series of the hypotenuses in BD as well as $B'D$ to write down an expression for $\sin 2\alpha$ in right-angled triangle ABD:

$$\sin(2\alpha) = \frac{BD}{AD} = \frac{2ab}{a^2 + b^2} \ \dots (2)$$

By equating equations 1 and 2 the students noted that $2ab$ cancels out on both sides, and by inverting both equations they obtained the desired result: $a^2 + b^2 = c^2$.

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[2] To prove the theorem of Pythagoras a more general definition of angles over a larger domain of say $(-360°; 360°)$ is not necessary as the other two angles in a right triangle are acute.

[3] Note that the construction of this diagram is based on the assumption that $\alpha > \beta$. For example, if $\beta > \alpha$ then right triangle ABD will lie to the other side of AB, and the perpendicular to BB' will have to be drawn at B to create the infinite series of similar right triangles. A more serious flaw of their proof is that when $\alpha = \beta = 45°$, then lines AD and BD are parallel, and no right triangle ABD is formed. The students needed to have considered this case separately, and it is not hard to prove Pythagoras in this case.

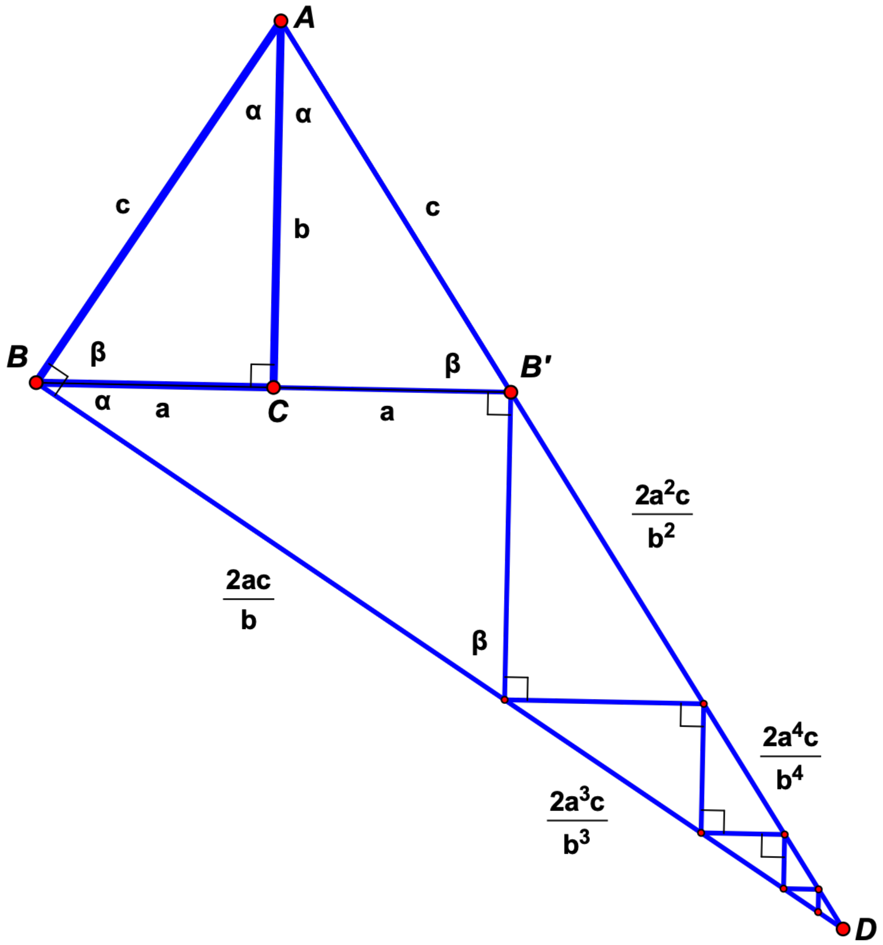

FIGURE 3

While the students' proof is quite ingenious, it is nonetheless also based on similarity, and one could easily rewrite it completely in terms of similarity without any reference to sine ratios. So although it is a rather more complicated proof than the first example, it is fundamentally the same type of 'trigonometric' proof.

UNIT CIRCLE DEFINITION OF TRIGONOMETRIC FUNCTIONS

All three examples of trigonometric proofs discussed in this article rely on the trigonometric ratios defined only for right-angled triangles – i.e. for positive acute angles, and not for obtuse, reflex or negative angles. While the ancient Greeks extended their application of trigonometry to obtuse angles in various ways[4], their approaches were limited in generality. From round about the Renaissance in the 17th century, as the need arose from various applications in science, the definition of the trigonometric ratios was extended to mathematical functions by using the so-called 'unit circle' definition of trigonometry.

This approach is well-known and is used in high schools around the world. Basically, we start off as shown in Figure 4 by considering a unit circle in the Cartesian plane with equation $x^2 + y^2 = 1$. Then $\cos\theta$ and $\sin\theta$ are respectively defined as the x- and y-coordinates of the point where the ray forming the angle θ intersects this circle.

[4] Ptolemy (100-170 AD), for example, used the theorem named after him (that for a cyclic quadrilateral $ABCD$, $AB \times CD + BC \times DA = AC \times BD$) to calculate sine values for angles between $90°$ and $180°$ (Maor, 1998).

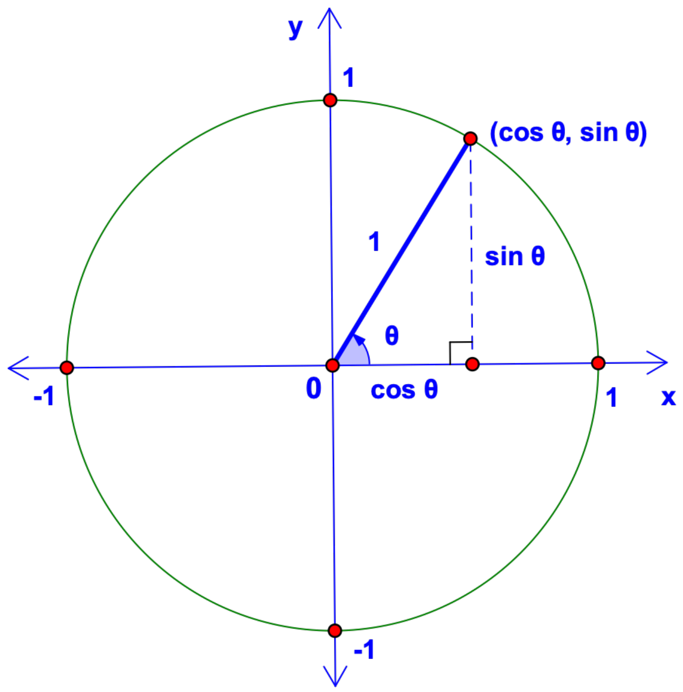

FIGURE 4

This is quite a brilliant and useful extension of the basic trigonometric ratios for right-angled triangles in order to be able to consistently handle obtuse, reflex or negative angles, as well as angles larger or smaller than $360°$ or $-360°$. However, with this gain there is also a loss. Not only is the distance between two points $(x_1; y_1)$ and $(x_2; y_2)$ in the Cartesian plane explicitly defined in terms of the Pythagorean theorem as $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, the use of the 'unit circle' definition also implicitly assumes the Pythagorean trigonometric identity $\sin^2 A + \cos^2 A = 1$ (as is clearly evident in Figure 4).

So, if one assumes the unit circle definitions of the trigonometric ratios as analytic/algebraic functions, then Loomis (and Dexter Luthuli) are perfectly correct in claiming that one cannot then produce a trigonometric proof for the theorem of Pythagoras, as that would automatically result in a circular argument!

CONCLUDING COMMENTS

To get back to the original question of whether a trigonometric proof for the theorem of Pythagoras is possible, the answer is unfortunately twofold: yes and no.

1) Yes, if we restrict the domain to positive acute angles, any valid similarity proof can be translated into a corresponding trigonometric one, or alternatively, we could use an approach like that of Zimba (2009) or Luzia (2015).

2) No, if we strictly adhere to the unit circle definitions of the trigonometric ratios as analytic functions, since that would lead to a circularity.

While trigonometric functions today can be defined in many different ways, for example by power series expansions, continued fractions, Euler's formula for the natural exponential function, etc., I am not aware of any approaches using these alternative definitions to prove the theorem of Pythagoras. However, this might be an interesting avenue for some of our readers to explore.

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