# A Cautionary Tale


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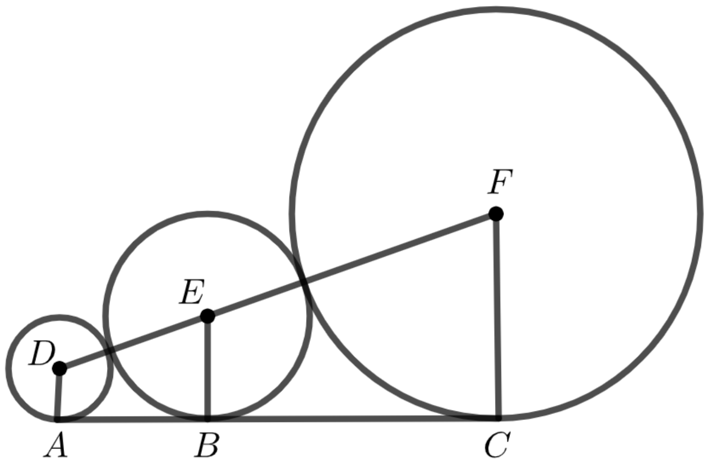
### INTRODUCTION

In this article we consider an externally sourced question that was "cut and pasted" directly into a recent Grade 12 Mathematics assessment. The question itself unfortunately contained a number of flaws. However, retrospectively unpacking these flaws, and reflecting on how one could suitably adjust the question, proved to be a meaningful learning experience.

The question is presented below, and the reader is encouraged to engage with the problem and to attempt to identify any flaws inherent in the presentation of the given scenario.

In the figure below, circles centre $D$, $E$ and $F$ have radii 3 cm, 5 cm and 7 cm respectively. The circles are tangent to one another and to the straight line $ABC$ at $A$, $B$ and $C$ as shown.




Given that $AB = 6$ cm, determine the length $BC$.

**FIGURE 1:** The original question.

### IDENTIFYING THE FLAWS

The reader will hopefully have identified a number of problems with the given diagram.

**Issue 1**

Since the two smaller circles have radii of 3 cm and 5 cm respectively, this means that $DE = 8$ cm. Since $DE$ has a relatively small angle of inclination, gut feel suggests that $AB$ is only going to be slightly less than 8 cm. The given length of 6 cm thus seems unlikely. Drawing a rectangle by adding a line from $D$ across to $EB$ (parallel to $AB$) allows us to calculate the actual length of $AB$ from the resulting right-angled triangle. We thus have $AB^2 = 8^2 - 2^2$ from which $AB = 7{,}75$ cm correct to two decimal places.

**Issue 2**

Although not explicitly stated, the diagram clearly suggests that $DEF$ is a straight line – this certainly seems to be the way it has been drawn. However, this is not the case. If we draw a line from $D$ perpendicular to $EB$, and another from $E$ perpendicular to $FC$, then we can use the two resulting right-angled triangles to determine the inclination of $DE$ and $EF$ relative to $ABC$:

- Inclination of $DE$: $\sin^{-1}\left(\frac{2}{8}\right) = 14{,}5°$
- Inclination of $EF$: $\sin^{-1}\left(\frac{2}{12}\right) = 9{,}6°$

This clearly shows that with the given radii, $DEF$ is *not* a straight line.

**Issue 3**

The question states that the three circles are tangential to one another as well as being tangential to the straight line $ABC$. This structure is simply not possible with the given radii – a fact that quickly becomes apparent if one tries to construct the image using geometry software such as GeoGebra. The necessary relationship between the three radii for this structure to be possible is explored more fully later in the article.

**Issue 4**

The inclusion of length $AB$ in the original question is a bit of a red herring since the length of $BC$ can be calculated by only considering the two larger circles. The circle with centre $D$ is irrelevant to this calculation. Drawing a rectangle by adding a line from $E$ across to $FC$ (parallel to $BC$), and using the resulting right-angled triangle, we have $BC^2 = 12^2 - 2^2$ from which $BC$ = 11,83 cm correct to two decimal places.

## REFLECTION

Students attempted to answer this question in a number of different ways, arriving at different values for the length of $BC$ depending on whether they used the spuriously given value of $AB$ = 6 cm and whether or not they assumed $DEF$ was a straight line. A number of students did both and made use of proportional reasoning such as $\frac{BC}{AB} = \frac{EF}{DE}$. Others extended $FD$ and $CA$ to a common point to form a large right-angled triangle (nesting two smaller right-angled triangles) and also made use of proportionality.

Interestingly, from a class of 23 only two of the strongest students commented on this particular question after the assessment, mentioning that they were left puzzled by the question and that they thought there might be something wrong with it. It was only during the subsequent feedback session where the given scenario was constructed in GeoGebra that most pupils picked up on the flaws inherent in the diagram as presented to them.

## THE GENERAL STRUCTURE

Let us consider a general scenario where we have three circles which are tangential to one another as illustrated in Figure 2. The three circles have centres $P$, $Q$ and $R$ and radii $r_1$, $r_2$ and $r_3$ respectively where $r_1 < r_2 < r_3$. The straight line $ABC$ is a tangent to each circle at $A$, $B$ and $C$ respectively. $PA$, $QB$ and $RC$ are perpendicular to $ABC$ (tangent perpendicular to radius). $PDE$ is parallel to $ABC$ with $D$ on $QB$ and $E$ on $RC$. $PQR$ is a straight line and $R\hat{P}E = \theta$.

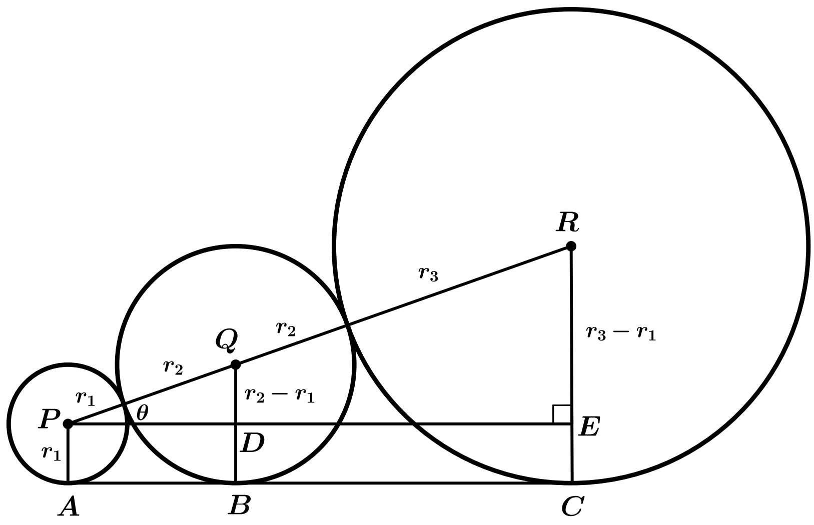



**FIGURE 2:** The general configuration.

With reference to Figure 2, using triangles $PQD$ and $PRE$, we can express $\sin\theta$ in two different ways:

$$\sin\theta = \frac{r_2 - r_1}{r_1 + r_2} = \frac{r_3 - r_1}{r_1 + 2r_2 + r_3}$$

From this it follows that:

$$(r_2 - r_1)(r_1 + 2r_2 + r_3) = (r_1 + r_2)(r_3 - r_1)$$

Simplifying the above leads to the following relationship between the three radii:

$$r_2{}^2 = r_1 r_3$$

The above result shows that if one wants all three radii to be whole numbers (or indeed rational numbers) then the product of $r_1$ and $r_3$ needs to be a perfect square. By way of example, if $r_1 = 2$ and $r_3 = 8$ then $r_2 = \sqrt{16} = 4$ and if $r_1 = \frac{18}{5}$ and $r_3 = \frac{72}{5}$ then $r_2 = \sqrt{\frac{1296}{25}} = \frac{36}{5}$. It should be clear that this is still valid in the trivial case of $r_1 = r_2 = r_3$.

In the given problem, $r_1 = 3$ and $r_3 = 7$, giving $r_1 r_3 = 21$ which is not a perfect square and therefore, with the given value of $r_2 = 5$, it is not possible for the circles to be tangential to one another and simultaneously tangential to the straight line $ABC$.

## CONCLUDING COMMENTS

It is unfortunate that the original flawed question found its way into an assessment without being fully checked. However, the subsequent process of identifying and unpacking the various flaws, and engaging with the mathematics behind how one could tweak the question so that it worked, proved to be a meaningful learning experience for students and teacher alike. Perhaps as an interesting classroom exercise one could present students with a flawed question and challenge them to identify the problems inherent in the question. This would certainly lead to some lively mathematical discussion and debate.