Squares Constructed on the Sides of an Acute-Angled Triangle

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INTRODUCTION

In this article we investigate the interesting geometrical set-up created by constructing squares on the sides of an acute-angled triangle as illustrated in Figure 1. There are a number of interesting results and invariant properties that one can develop from this starting point, and these have the potential to be used as classroom explorations or directed investigations. For the purposes of this article, we will largely focus on results that relate to areas.

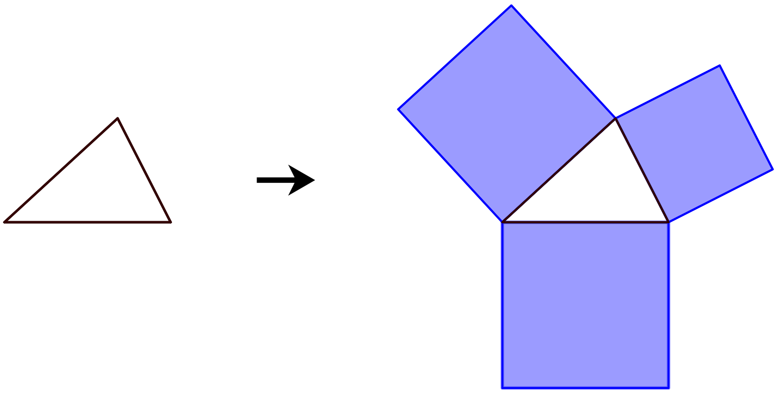

FIGURE 1: Squares constructed on the sides of an acute-angled triangle.

INTERESTING AREA SUBDIVISION INVARIANCE

Begin with acute-angled triangle ABC with a square constructed on each side and place an arbitrary point P inside the triangle. From point P construct perpendiculars to each side of the triangle. Extend each of these perpendiculars to divide each square into two rectangles as illustrated in Figure 2.

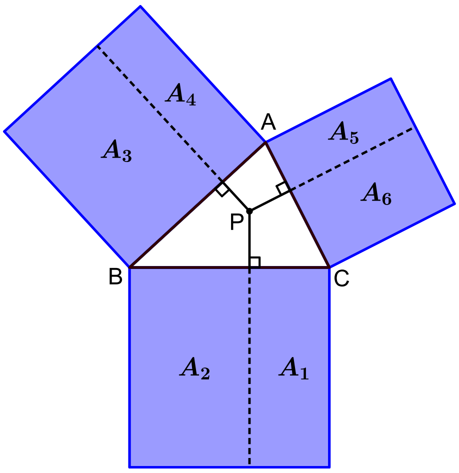

FIGURE 2: Subdividing the squares using an arbitrary point P in the triangle.

Labelling the areas of these rectangles as A_1 to A_6 in a circular fashion, then the following interesting invariant property holds with respect to the six areas:

$$A_1 + A_3 + A_5 = A_2 + A_4 + A_6$$

We can prove this result as follows, with reference to Figure 3.

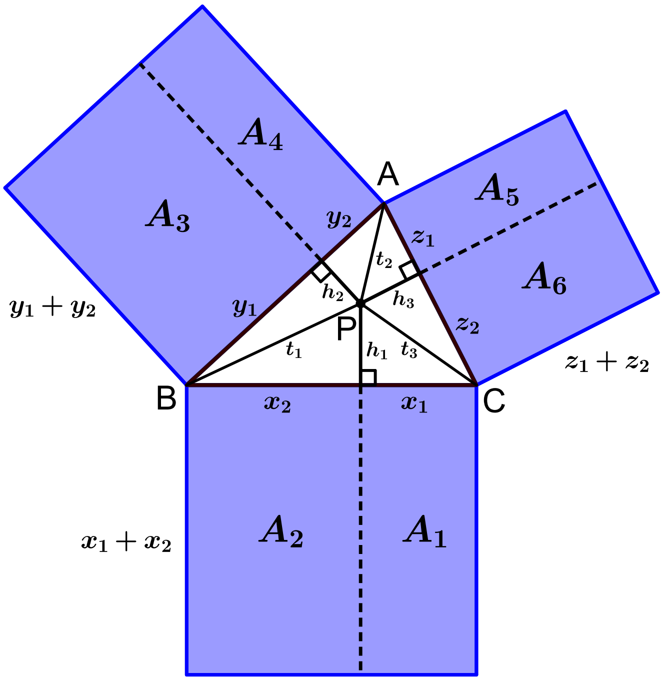

FIGURE 3: Proving that $A_1 + A_3 + A_5 = A_2 + A_4 + A_6$.

$$A_1 + A_3 + A_5 = x_1(x_1 + x_2) + y_1(y_1 + y_2) + z_1(z_1 + z_2) = {x_1}^2 + {y_1}^2 + {z_1}^2 + x_1x_2 + y_1y_2 + z_1z_2$$
$$A_2 + A_4 + A_6 = x_2(x_1 + x_2) + y_2(y_1 + y_2) + z_2(z_1 + z_2) = {x_2}^2 + {y_2}^2 + {z_2}^2 + x_1x_2 + y_1y_2 + z_1z_2$$

In order to prove our result, all we now need to do is show that ${x_1}^2 + {y_1}^2 + {z_1}^2 = {x_2}^2 + {y_2}^2 + {z_2}^2$. In Figure 3 we have included line segments connecting point P to the vertices of triangle ABC. These have been labelled t_1, t_2 and t_3. The three altitudes have been labelled h_1, h_2 and h_3. We can now use the Pythagorean theorem in each of the six small right-angled triangles into which triangle ABC is subdivided.

$$\begin{aligned}
{t_1}^2 &= {x_2}^2 + {h_1}^2 &\rightarrow\quad {x_2}^2 &= {t_1}^2 - {h_1}^2 \\
{t_1}^2 &= {y_1}^2 + {h_2}^2 &\rightarrow\quad {y_1}^2 &= {t_1}^2 - {h_2}^2 \\
{t_2}^2 &= {y_2}^2 + {h_2}^2 &\rightarrow\quad {y_2}^2 &= {t_2}^2 - {h_2}^2 \\
{t_2}^2 &= {z_1}^2 + {h_3}^2 &\rightarrow\quad {z_1}^2 &= {t_2}^2 - {h_3}^2 \\
{t_3}^2 &= {z_2}^2 + {h_3}^2 &\rightarrow\quad {z_2}^2 &= {t_3}^2 - {h_3}^2 \\
{t_3}^2 &= {x_1}^2 + {h_1}^2 &\rightarrow\quad {x_1}^2 &= {t_3}^2 - {h_1}^2
\end{aligned}$$

We thus have:

$$\begin{aligned}
{x_1}^2 + {y_1}^2 + {z_1}^2 &= {t_1}^2 + {t_2}^2 + {t_3}^2 - \left({h_1}^2 + {h_2}^2 + {h_3}^2\right) \\
{x_2}^2 + {y_2}^2 + {z_2}^2 &= {t_1}^2 + {t_2}^2 + {t_3}^2 - \left({h_1}^2 + {h_2}^2 + {h_3}^2\right)
\end{aligned}$$

This shows that ${x_1}^2 + {y_1}^2 + {z_1}^2 = {x_2}^2 + {y_2}^2 + {z_2}^2$ and completes the proof. This latter result is known as Carnot's perpendicularity theorem, named after the French mathematician Lazare Carnot (1753-1823).

Triangles formed by the squares

We once again begin with acute-angled triangle ABC with a square constructed on each side. The vertices of the squares are then connected as illustrated in Figure 4. The interesting result is that each of the triangles constructed in this fashion has the same area as the original triangle ABC, i.e. $A_1 = A_2 = A_3 = A_4$.

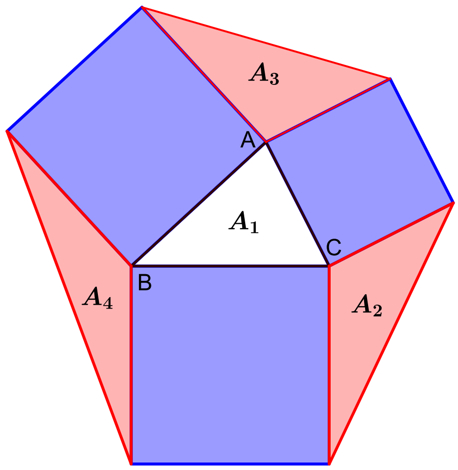

Figure 4: Forming three further triangles of equal area to ΔABC.

This result is historically attributed to Vecten, a 19th century French mathematician. There are a number of ways that one can prove this result. We will consider three different approaches – one using trigonometry, one using geometric constructions, and one involving dynamic geometric visualisation.

Proof 1

With reference to Figure 5:

$$A_2 = \frac{1}{2}ab\sin(180° - \theta) = \frac{1}{2}ab\sin\theta = A_1$$
$$A_3 = \frac{1}{2}bc\sin(180° - \alpha) = \frac{1}{2}bc\sin\alpha = A_1$$
$$A_4 = \frac{1}{2}ac\sin(180° - \beta) = \frac{1}{2}ac\sin\beta = A_1$$

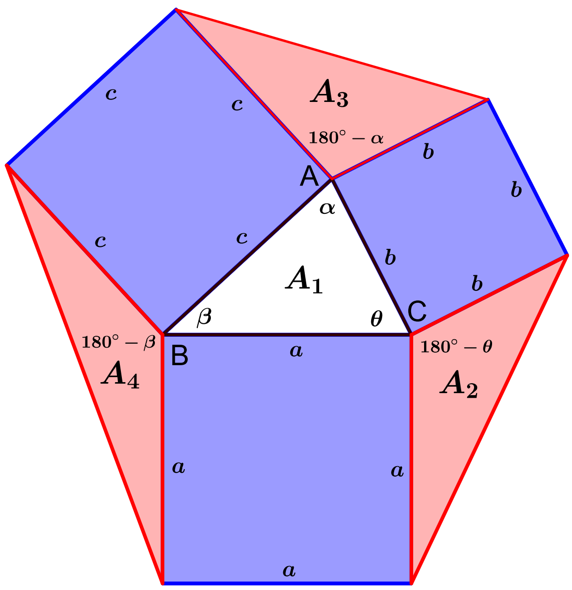

Figure 5: A trigonometric proof.

Proof 2

With reference to Figure 6, extend one side of each square to form three new triangles, each of which is congruent to the original triangle ABC. Each of these triangles has the same area as its adjoining triangle since the median of the large triangle (formed by each pair of adjoining triangles) bisects the area of the large triangle, from which it follows that $A_1 = A_2 = A_3 = A_4$.

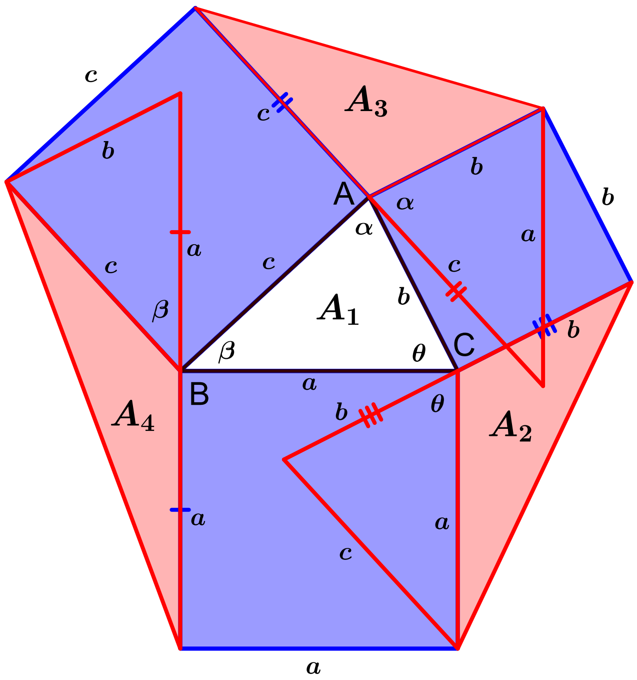

FIGURE 6: A proof using geometric construction and congruent triangles.

Proof 3

The third proof, as illustrated in Figure 7, relies on dynamic geometric visualisation and is attributed as a 'proof without words' to Steven L. Snover. The proof again relies on the observation that the median of a triangle subdivides the triangle into two smaller triangles of equal area. Expressed differently, the median of a triangle subdivides the triangle into two smaller triangles each having the same length base and the same perpendicular height, and hence the same area.

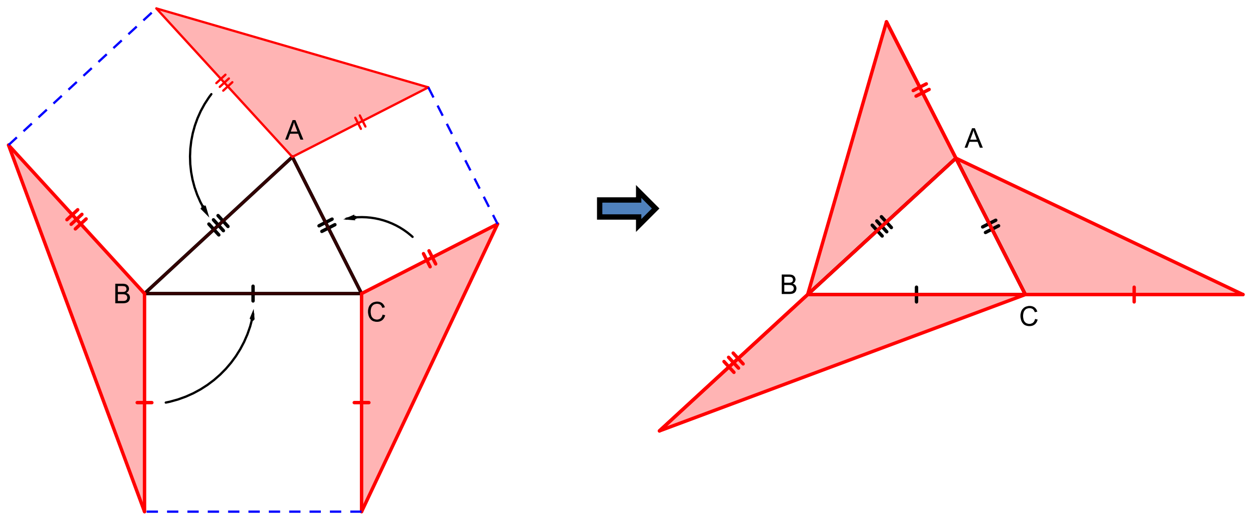

FIGURE 7: A proof without words relying on dynamic geometric visualisation.

AN INTERESTING GEOMETRIC EXTENSION

Let us now return to the original configuration. A further interesting result (although not related to area) arises when each of the medians of the central triangle ABC is extended to the opposite side of the outer triangle through its connecting vertex, as illustrated in Figure 8. The interesting result is that each of these extended medians is perpendicular to the opposite side of its connected triangle.

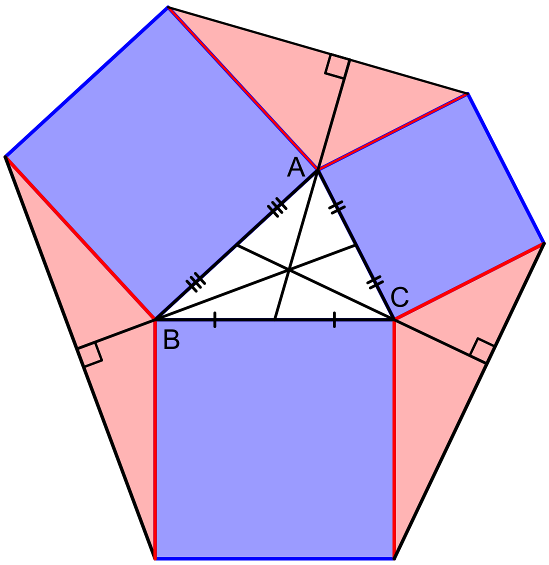

FIGURE 8: Drawing and extending the three medians of triangle ABC.

This result can readily be proved using an elegant construction as illustrated in Figure 9. Use the original triangle ABC and construct parallelogram $ABDC$. Since co-interior angles on parallel lines are supplementary, we have $A\hat{C}D = 180° - (\alpha + \beta)$, and from angles round a point we have $E\hat{A}H = 180° - (\alpha + \beta)$. From this it follows that ΔEAH and ΔDCA are congruent (SAS). Thus $A\widehat{H}F = \beta$, and since $F\hat{A}H = 90° - \beta$ (angles on a straight line) it follows that $A\hat{F}H = 90°$, which proves the result.

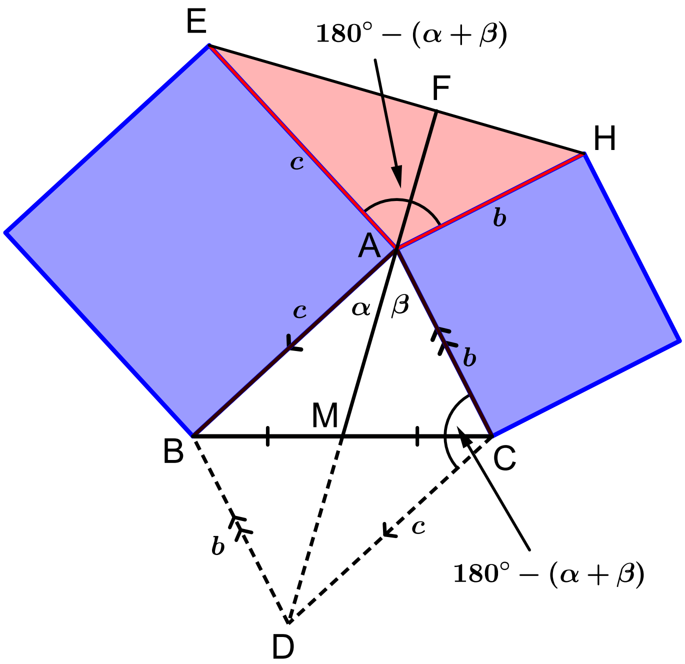

FIGURE 9: Proving the result with an elegant construction.

A FINAL EXTENSION INVESTIGATION

Returning to Figure 4, consider what would happen if squares were constructed on the third face of each of the three outer triangles, and these were then connected to form quadrilaterals as illustrated in Figure 10.

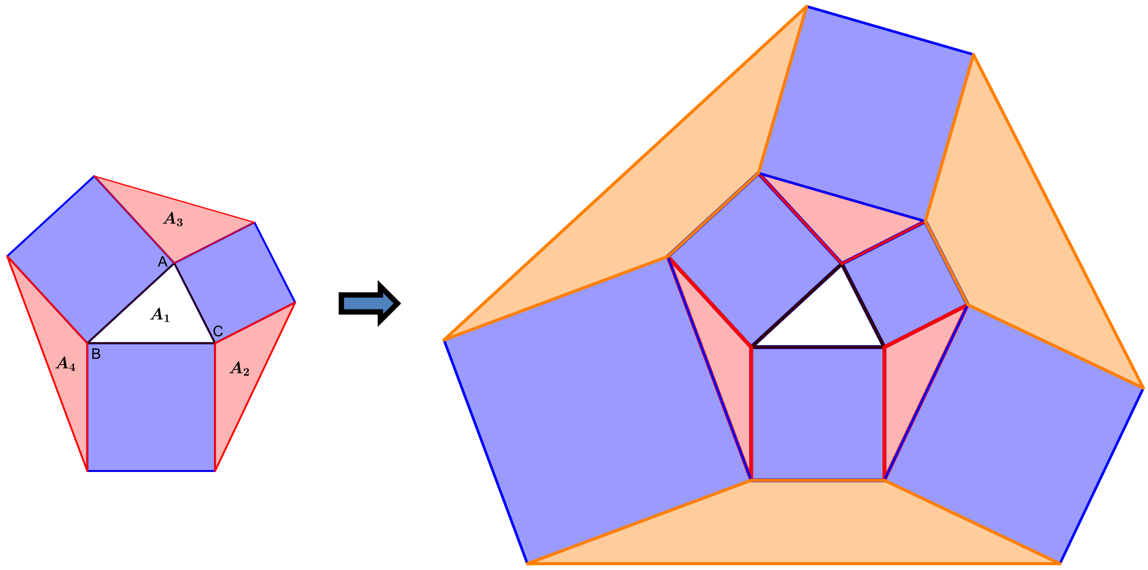

FIGURE 10: Extending the structure outwards.

As an initial observation, note that each of the quadrilaterals formed is a trapezium, i.e. each quadrilateral has a pair of opposite sides parallel. Furthermore, the area of each of these trapeziums is the same, and this area is exactly five times the area of the original central triangle. It is left to the interested reader to explore this further.

CONCLUDING COMMENTS

The purpose of this article was to investigate an interesting geometrical configuration – squares constructed on the sides of an acute-angled triangle – and to highlight a number of interesting results and invariant properties that could be explored using this specific geometrical configuration as a starting point. Many of these have the potential to be worked into classroom explorations or directed investigations.

For the purposes of this article we largely focused on results relating specifically to area, but there are many other geometrical results that could be explored using this particular configuration as a starting point. Furthermore, we have specifically focused on an acute-angled central triangle. Do these results still hold true in the case of an obtuse-angled triangle? What if, instead of constructing the three squares outwardly, we constructed them inwardly so that they overlapped? More generally, instead of starting with a central triangle, do these results still hold true if we start with a different convex polygon such as a quadrilateral, pentagon or hexagon? These further considerations are left to the interested reader to explore.

The original configuration of squares constructed on the sides of a triangle is known variously as Vecten's configuration or the *Bride's Chair* and is a well-known geometric configuration. It has been studied quite extensively, illustrating the immense richness of this simple configuration. For readers who wish to explore this further, the following links are suggested:

https://www.cut-the-knot.org/ctk/BridesChair.shtml

https://en.wikipedia.org/wiki/Vecten_points

http://dynamicmathematicslearning.com/bottema.html

http://dynamicmathematicslearning.com/crossdiscovery.html