# Series Summation – An Extension of a 1972 Matric Examination Question


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### INTRODUCTION

While preparing for my Matric Mathematics examination in 1973, included in my revision was the following question from the previous year's Matric examination:

> Given the series $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots$ , determine the general formula in terms of $x$ for $S_{40}$, the sum of the first $40$ terms.

Rather than determining a general formula for $S_{40}$, this article looks at determining a general formula in terms of $x$ ($x \in \mathbb{R}$) for $S_n$, the sum of the first $n$ terms. Readers are encouraged to attempt this before reading on. Once the general formula has been established, a particular value of $x$ is then considered.

### DETERMINING THE GENERAL FORMULA

We have $S_n = 1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{n-1}$. Although there is an easily discernible pattern to the sequence of terms, the series itself is neither arithmetic nor geometric, which precludes us from using the standard $S_n$ formulas. However, with some clever manipulation we can get around this problem quite elegantly.

### METHOD 1

Begin with the statement $S_n = 1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{n-1}$ and multiply each side by $(1-x)$. This has the effect of creating a 'collapsing' or 'telescoping' series:

$$\begin{aligned}(1-x)S_n &= 1(1-x) + 2x(1-x) + 3x^2(1-x) + 4x^3(1-x) + \ldots + nx^{n-1}(1-x) \\ &= 1 - x + 2x - 2x^2 + 3x^2 - 3x^3 + 4x^3 - 4x^4 + \ldots + nx^{n-1} - nx^n \\ &= (1 + x + x^2 + x^3 + \ldots + x^{n-1}) - nx^n\end{aligned}$$

Since the $n$ bracketed terms represent a geometric series with $a = 1$ and $r = x$ we can now use the standard formula for the sum of a geometric series:

$$(1-x)S_n = \frac{1(1-x^n)}{1-x} - nx^n$$

Combining the right-hand side and dividing through by $1-x$ gives:

$$S_n = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2} \quad ; \quad x \neq 1$$

**METHOD 2**

An alternative manipulation is to split each term, with the exception of the first, as follows:

$$S_n = 1+2x+3x^2+4x^3+\ldots+nx^{n-1}$$

$$S_n = 1+(x+x)+\left(x^2+2x^2\right)+\left(x^3+3x^3\right)+\ldots+\left(x^{n-1}+(n-1)x^{n-1}\right)$$

We can now re-group as follows:

$$S_n = \left(1+x+x^2+x^3+\ldots+x^{n-1}\right)+\left(x+2x^2+3x^3+\ldots+(n-1)x^{n-1}\right)$$

$$= \left(1+x+x^2+x^3+\ldots+x^{n-1}\right)+x\left(1+2x+3x^2+\ldots+(n-1)x^{n-2}\right)$$

The first set of bracketed terms is a geometric series with $a=1$ and $r=x$, while the second set of bracketed terms is simply $S_n - nx^{n-1}$. We thus have:

$$S_n = \frac{1(1-x^n)}{1-x}+x\left(S_n-nx^{n-1}\right)$$

This can be rearranged to give $S_n - xS_n = \dfrac{1-x^n-nx^n+nx^{n+1}}{1-x}$, from which we can write our final formula:

$$S_n = \frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2} \quad ; \quad x \neq 1$$

**SPECIAL CASE: $x = 1$**

Since our formula does not hold for $x=1$ we need to consider this as a special case. When $x=1$ we have:

$$S_n = 1+2+3+4+\ldots+n$$

This is a simple arithmetic sequence of $n$ terms with $a=1$ and $d=1$. Using the standard formula for the sum of an arithmetic series yields:

$$S_n = \frac{n}{2}(1+n)$$

The full general formula for $S_n$ is thus:

$$S_n = \frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2} \;;\; x \neq 1 \quad \text{or} \quad S_n = \frac{n}{2}(1+n) \text{ for } x=1$$

**DISCUSSION OF THE SERIES WHEN $x = -1$**

When $x=-1$, then the series becomes $S_n = 1-2+3-4+5-6+\ldots\pm n$. Note that the final term has been expressed as $\pm n$ since it would be $+n$ for an odd number of terms and $-n$ for an even number of terms. Taken as an isolated question, i.e. without consideration of the previous discussion, determining the general formula for the sum of this series would certainly be considered a non-routine question. Readers are invited to attempt this before reading on.

If we group terms in pairs, then we have:

$$S_n = (1-2)+(3-4)+(5-6)+\ldots$$

Since each grouped pair sums to $-1$, if $n$ is even there will be $\frac{n}{2}$ pairs of $-1$. Thus:

$$S_n = -\frac{n}{2}$$

However, if $n$ is odd, then there will be $\frac{n-1}{2}$ pairs, each summing to $-1$, plus a lone final term of $+n$. Thus:

$$S_n = -\left(\frac{n-1}{2}\right)+n = \frac{-n+1+2n}{2} = \frac{n+1}{2}$$

Although we have established a general formula for the sum of the series for each of the two possible scenarios, i.e. when $n$ is even and when $n$ is odd, it would be pleasing to be able to combine these into a single general formula. In order to do accomplish this we can make use of the expression $[1-(-1)^{n+1}]$, which yields a value of $2$ when $n$ is even and a value of $0$ when $n$ is odd, along with the expression $[1-(-1)^n]$, which yields a value of $0$ when $n$ is even and a value of $2$ when $n$ is odd. Making use of these two expressions we can create a single general formula for the sum of the series:

$$S_n = -\frac{n}{2}\times\frac{[1-(-1)^{n+1}]}{2} + \frac{n+1}{2}\times\frac{[1-(-1)^n]}{2}$$

If $n$ is even then the second term in the above formula collapses to zero, and the first term simplifies to give $S_n = -\frac{n}{2}$. If $n$ is odd then the first term in the formula collapses to zero, and the second term simplifies to give $S_n = \frac{n+1}{2}$. As a final step we can combine the two terms in the formula and simplify the algebra to give:

$$S_n = \frac{1+(-1)^{n+1}(2n+1)}{4}$$

## CONCLUDING REMARKS

The use of non-routine questions is an important element in facilitating understanding of the mathematical concepts being taught, and indeed often leads to further investigative opportunities. Consider for example the above formula for $S_n$. Since we know that the sum of the series is an integer value, it follows that the expression $1+(-1)^{n+1}(2n+1)$ must be divisible by 4 for all $n \in \mathbb{N}$, which is certainly not intuitive in isolation. The veracity of this observation can readily be verified through mathematical induction.