Packing Four Spheres into a Tetrahedron

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A few years ago, when learning about 3-dimensional shapes, my Grade 9 Maths class was tasked with building containers to hold four ping pong balls of radius 2 cm so that the four balls fitted snugly inside the container. The simplest and most popular ways of doing this were to construct cuboids or cylinders with the correct dimensions. If you were ambitious you could try other shapes, such as a triangular prism, but whatever shape you chose it was important for the dimensions to be mathematically accurate. Calculating the dimensions of these shapes is well within the means of a competent Grade 9 learner.

Since the container had to hold four balls, a natural shape to think of for this task is the tetrahedron (Figure 1). Calculating the required side length for a tetrahedral container is far more complicated, and certainly not within the capabilities of a Grade 9 learner. Although this is the shape I chose for the container, and although I managed to construct a reasonably accurate tetrahedron, the dimensions were not precise. Having being reminded of the container activity I decided to set out to solve the problem precisely and find the exact measurement for the side length of the tetrahedral container. To fully appreciate the problem I urge readers to attempt it first, as it requires a high degree of visualisation and is trickier than it might seem.

Suppose we wish to pack four spheres of radius r in a tetrahedron. Let the side length of the required tetrahedron be x. Let us first consider a cross-section where one of the spheres meets two of the sides of the tetrahedron (Figure 2)

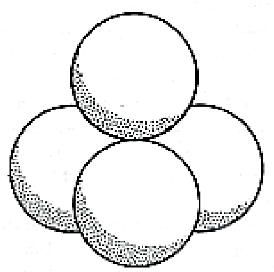

FIGURE 1

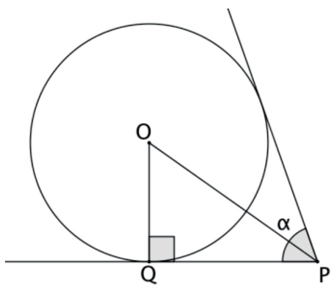

FIGURE 2

Before we can do anything with this we first need to know the size of angle α. Note that α is simply the angle for the slope of the face relative to the 'base', i.e. the angle between two adjacent faces of the tetrahedron. In order to calculate this let us imagine the plane that 'splits' the tetrahedron by passing through an edge as well as the midpoint of the opposite edge (Figure 3). The isosceles triangle that lies on this plane is illustrated in Figure 4.

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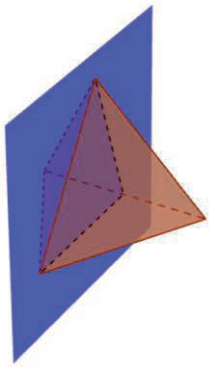

FIGURE 3

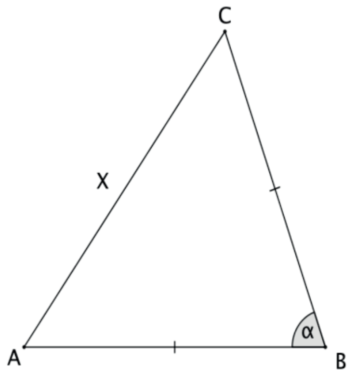

FIGURE 4

Keeping in mind that each tetrahedral face is an equilateral triangle of side length x, we can see that AB and BC are the perpendicular heights of two of these faces. Since AB and BC are perpendicular heights of an equilateral triangle we have $\mathrm{AC} = x$ and $\mathrm{AB} = \mathrm{BC} = \frac{\sqrt{3}}{2}x$. We can now use the cosine rule to calculate α:

$$\left(\frac{\sqrt{3}}{2}x\right)^2 + \left(\frac{\sqrt{3}}{2}x\right)^2 - 2\left(\frac{\sqrt{3}}{2}x\right)^2 \cos\alpha = x^2$$

$$\therefore \ \frac{3}{2}x^2(1 - \cos\alpha) = x^2$$

$$\therefore \ \cos\alpha = \frac{1}{3}$$

We thus have $\alpha = 70{,}53°$ (to two decimal places). Referring back to Figure 2, since $\mathrm{O\hat{P}Q} = \frac{\alpha}{2}$ and $\mathrm{OQ} = r$ we can express the length of QP as $\mathrm{QP} = \frac{\mathrm{OQ}}{\tan\left(\frac{\alpha}{2}\right)} = 1{,}41r$ (to two decimal places). We now have all the information we need to relate x (the side length of the tetrahedron) to r (the radius of the sphere).

For this final step we need to look at the triangular 'base' of the tetrahedron. Note that three of the spheres touch the base, and these are indicated by the vertices of the central triangle in Figure 5. Since F and G are points at which a sphere touches the base, we have $\mathrm{FG} = \mathrm{HI} = 2r$. Note that FI and GH are both equal in length to QP (see Figure 2) which we have already calculated to be $1{,}41r$. We also have $\mathrm{IE} = \mathrm{DH} = \frac{\mathrm{FI}}{\tan 30°} = 2{,}45r$. Finally, we can write $x = \mathrm{DH} + \mathrm{HI} + \mathrm{IE} = 6{,}899r$, or expressed more accurately in surd form, $x = \left(2 + 2\sqrt{6}\right)r$. Deriving this surd form is left as an exercise to the reader.

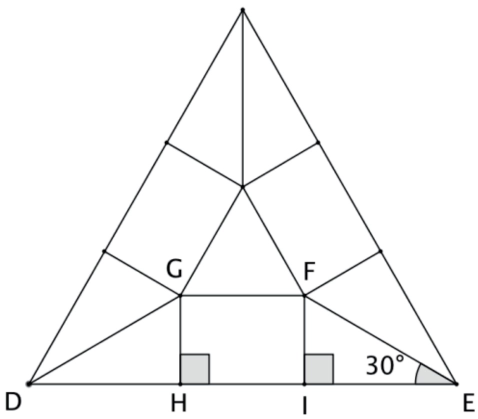

FIGURE 5