Finding the Maximum Distance between Two Curves

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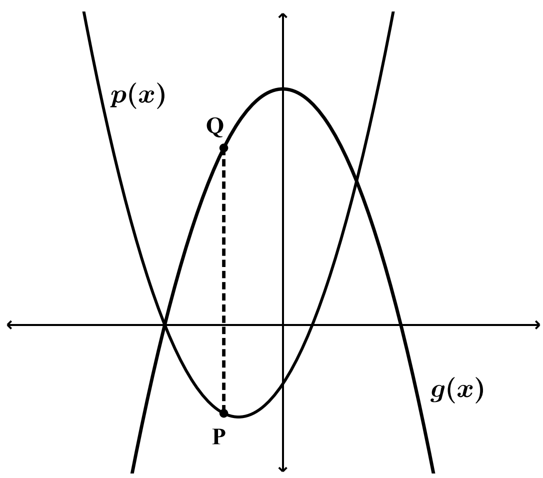
Nothing ever becomes real until it is experienced – John Keats

Consider the graphs of the functions $p(x) = x^2 + \frac{3}{2}x - 1$ and $g(x) = -x^2 + 4$. P is a point on p and Q is a point on g such that QP is parallel to the y-axis. What is the maximum length of QP between the points of intersection of the two graphs?

This is a fairly standard question that can be answered using a number of different approaches. The length of QP is given by $g(x) - p(x)$ which simplifies to:

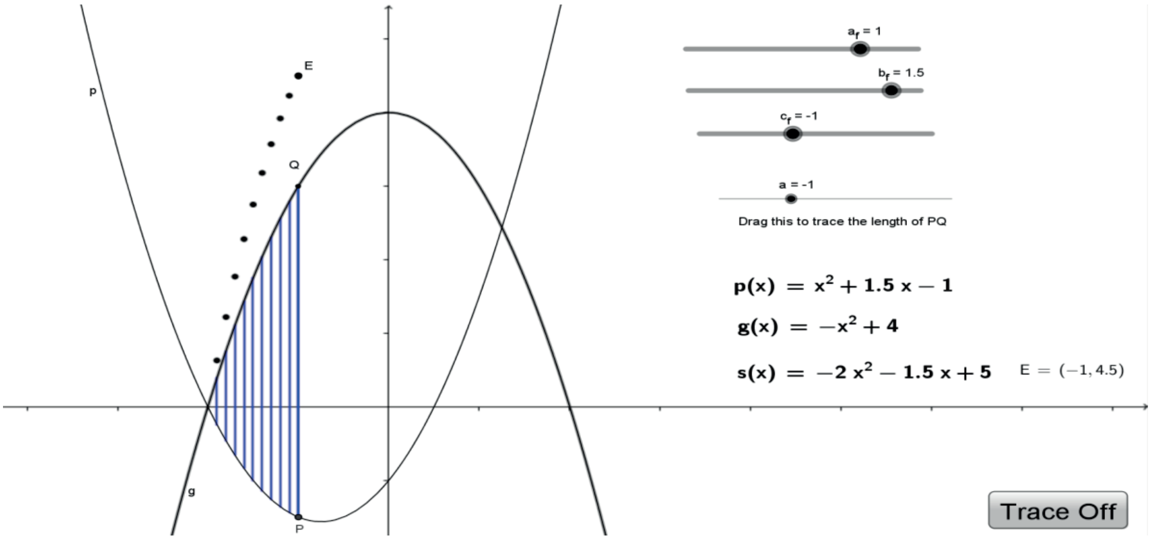
$$L(x) = -2x^2 - \frac{3}{2}x + 5$$



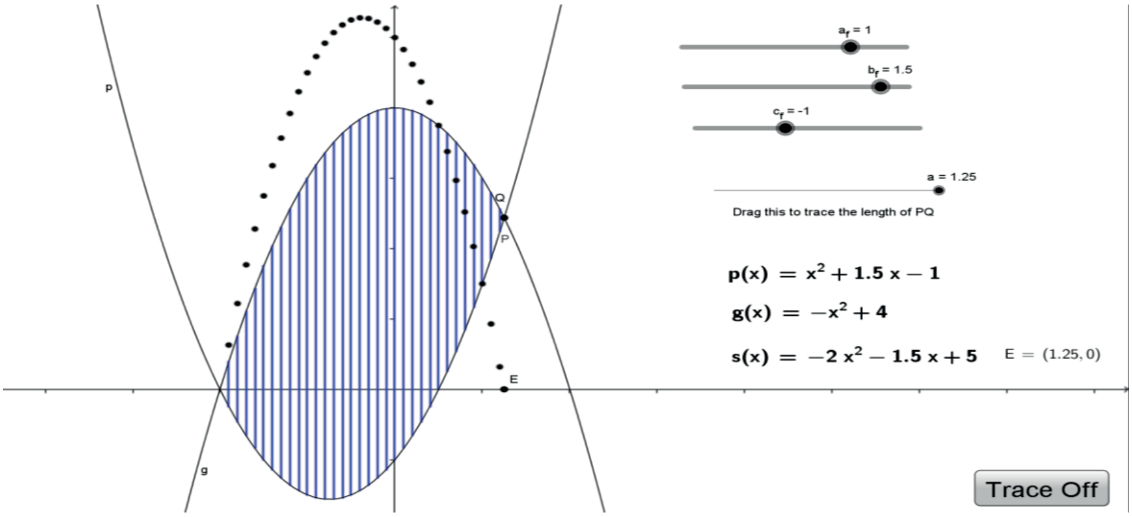
The points of intersection of p and g can be found by setting $L(x) = 0$ since the distance QP will be zero at these points. Solving $L(x) = 0$ gives $x = -2$ and $x = \frac{5}{4}$. In order to determine the maximum length of QP all we need to do is determine the turning point of the parabola represented by $L(x)$.

We can do this by (i) completing the square and writing $L(x)$ in the form $a(x-p)^2 + q$ from which the turning point can be read, (ii) determining the axis of symmetry of L and using it to find the corresponding y-value, or (iii) setting $L'(x) = 0$ to determine the point at which the gradient is zero and then finding the corresponding y-value. Using any of these methods gives the turning point of L to be $\left(-\frac{3}{8}; \frac{169}{32}\right)$. The maximum length of QP is thus $5{,}28$ to two decimal places.

Irrespective of the approach taken, the most critical aspect is for pupils to appreciate that the expression $L(x) = g(x) - p(x)$ represents the *length* of the line segment QP. The use of dynamic geometry software such as *GeoGebra* is a useful way of adding a more visual element to reinforce this. We began by drawing the two original parabolas, $p(x) = x^2 + \frac{3}{2}x - 1$ and $g(x) = -x^2 + 4$. The distance PQ was then defined by the function $s(x) = -2x^2 - \frac{3}{2}x + 5$. We then set E to be a point on s between $x = -2$ and $x = \frac{5}{4}$, i.e. between the points of intersection of p and g. A slider was then created so that the x-value of point E could be adjusted for $x \in \left[-2; \frac{5}{4}\right]$. The file was set up so that as the slider is moved it not only traces the curve of $g(x) - p(x)$ but also the line segment QP.



As pupils engage with the dynamic geometry sketch their attention should be drawn to a number of important visual elements: (i) the traced function represented by $g(x) - p(x)$ has x-intercepts at the x-values where $g(x)$ and $p(x)$ intersect, (ii) as the slider is moved from $x = -2$ to $x = \frac{5}{4}$ the length of QP increases to a maximum and then decreases again, (iii) the maximum length of QP coincides with the point at which the function represented by $g(x) - p(x)$ has a local maximum.



The use of a dynamic geometry environment to reinforce visual elements of what is otherwise a fairly abstract algebraic process can be incredibly powerful. For those interested, we have uploaded the GeoGebra file to http://tube.geogebra.org/ under the title "Distance Between Parabolas". Additional sliders have been added so that the equation of $p(x)$ can also be adjusted.