A Multiple Solution Task: A South African Mathematics Olympiad Problem

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INTRODUCTION

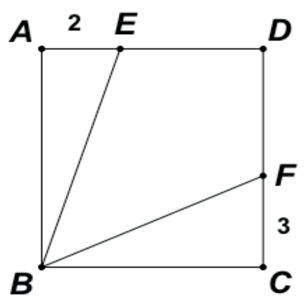
Exploring and engaging with multiple solution tasks (MSTs), where students are given rich mathematical tasks and encouraged to find multiple solutions (or proofs), has been an interesting and productive trend in problem solving research in recent years. A longitudinal comparative study by Levav-Waynberg and Leikin (2012) suggests that an MST approach in the classroom provides a greater educational opportunity for potentially creative students when compared with a conventional learning environment.

Reflection and discussion in the classroom of MST problems can be a very useful learning experience for developing young mathematicians. George Polya (1945) specifically mentions the consideration of alternative solutions or proofs in the final stage of his model for problem solving, namely 'Looking Back'. The value of this process of reflection is not only to better understand the problem by viewing it from different perspectives, but also to encourage flexibility of thought and to enhance one's repertoire of problem solving skills and approaches for future challenges.

Although the South African Mathematics Olympiad (SAMO) doesn't explicitly ask students to arrive at multiple solutions for problems (nor does it provide an opportunity for students to write down their full solutions in the first two rounds), there are usually several problems in the first two rounds that can be solved and proved in numerous ways. In fact the potential for a problem to be solved in multiple ways is frequently used as a selection criterion, especially for harder problems.

THE PROBLEM

Let us now look at one such MST example – Question 20 from the Senior Section First Round paper of the 2016 SAMO[2].



If $ABCD$ is a square, $E\hat{B}F = C\hat{B}F$, $AE = 2$ and $CF = 3$, then the length of EB is

(A) $\sqrt{13}$ (B) 5 (C) 6 (D) $\sqrt{29}$ (E) $4\sqrt{3}$

[2] Past papers of the SAMO are available at: http://www.samf.ac.za/QuestionPapers.aspx

Underlying this problem is the following interesting and more general theorem: If $ABCD$ is a square, and an arbitrary point E is chosen on AD and $E\hat{B}C$ is bisected by BF, with F on CD, then $EB = AE + CF$. Before continuing, readers are encouraged first to dynamically explore the general theorem using the interactive sketch at http://dynamicmathematicslearning.com/samo2016-R1Q20.html and attempt to prove the result themselves[3].

SOLUTIONS AND PROOFS

Three official solutions were given for Question 20, each using a different approach and different techniques.

SOLUTION 1: USING TRIGONOMETRY

If we let $E\hat{B}F = C\hat{B}F = \theta$, then $A\hat{E}B = 2\theta$ (alternate angles, $AD \parallel BC$). If x denotes the length of the side of the square, then $\tan\theta = \frac{3}{x}$ from triangle BCF and $\tan 2\theta = \frac{x}{2}$ from triangle BEA. From the double-angle formulae, $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$, it is easy to show that if $t = \tan\theta$ then $\tan 2\theta = \frac{2t}{1-t^2}$. We thus have:

$$\frac{x}{2} = \frac{\frac{6}{x}}{1-\left(\frac{3}{x}\right)^2} = \frac{6x}{x^2-9}$$

This simplifies to $x^2 - 9 = 12$ (since $x \neq 0$), so $x^2 = 21$. Finally, by Pythagoras' Theorem:

$$BE^2 = AB^2 + AE^2 = x^2 + 4 = 21 + 4 = 25$$

Thus $BE = 5$.

SOLUTION 2: USING SIMILAR TRIANGLES

It is possible to solve the problem using only similar triangles. Let G be the point on EB such that $EG = EA$ and let H be the foot of the perpendicular from G to AB, then triangles BHG and BAE are similar. Also, $E\hat{A}G = 90° - \theta$, since triangle EAG is isosceles, so $G\hat{A}H = \theta$ and triangle AHG is similar to triangle BCF. Using these facts we have:

$$\frac{GB}{EG} = \frac{HB}{AH} = \frac{HB}{HG} \times \frac{HG}{AH} = \frac{AB}{AE} \times \frac{CF}{BC} = \frac{CF}{AE} \text{ (since } AB = BC\text{)}$$

It follows that $GB = CF$, since $EG = AE$ by construction, and finally $EB = EG + GB = AE + CF = 5$.

SOLUTION 3: USING A 90° ROTATION

An even quicker and more elegant solution is the following. Rotate triangle ABE through 90° clockwise around B so that A coincides with C and E is at position E' on DC produced. In other words, let E' be the point on DC produced such that $CE' = AE$, and join B and E' so that triangles BAE and BCE' are congruent. Then $E'\hat{F}B = 90° - \theta$ and $E'\hat{B}F = E'\hat{B}C + C\hat{B}F = (90° - 2\theta) + \theta = 90° - \theta$. Thus triangle $E'BF$ is isosceles, and therefore $E'B = E'F$. Since $E'B = EB$ and $E'F = E'C + CF = AE + CF$ it follows that $EB = AE + CF = 5$.

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[3] As an interesting aside, if one makes use of directed distances, i.e. if distances are treated as vectors, then the result remains true even if E falls on the *extension* (both sides) of AD.

SOLUTION 4: USING AREAS

After the paper was written, the SAMO committee also received the following neat alternative solution from Nicholas Kroon in Grade 12 at St Andrew's College in Grahamstown:

Let $E\hat{B}F = F\hat{B}C = \theta$ and let x denote the length of the side of the square. We can now calculate the area of the square in two different ways:

$$[ABCD] = x^2 = [BAE] + [BFC] + [EDF] + [BEF]$$

Using Pythagoras' Theorem and the sine area rule we then have:

$$x^2 = x + \frac{3}{2}x + \frac{(x-2)(x-3)}{2} + \frac{1}{2}\sqrt{x^2+4} \times \sqrt{x^2+9} \times \sin\theta$$

Note that from triangle BFC we know that $\sin\theta = \frac{3}{\sqrt{x^2+9}}$. Hence:

$$x^2 = \frac{5}{2}x + \frac{(x-2)(x-3)}{2} + \frac{1}{2}\sqrt{x^2+4} \times 3$$

$$\Rightarrow \; 2x^2 = x^2 + 6 + 3\sqrt{x^2+4}$$

$$\Rightarrow \; (x^2-6)^2 = 9x^2 + 36$$

$$\Rightarrow \; x^2(x^2-21) = 0$$

Note that since $x > 0$ it follows that $x = \sqrt{21}$ and hence, using Pythagoras' Theorem, $EB = 5$.

CONCLUDING COMMENTS

It is left to the reader to compare the pros and cons of each solution, but in particular to note that the general theorem for a square and an arbitrarily chosen point E on AD is most easily seen and immediately generalizable from proofs 2 and 3. It can also be seen from proof 3 that attempting a similar rotation of triangle ABE for a rectangle or a rhombus won't preserve the general relationship $EB = AE + CF$.

It would be wonderful if more pupils, and also teachers, submitted interesting alternative solutions for SAMO questions, as it is likely that the official solutions are often not the only ones.

REFERENCES

Levav-Waynberg, A., & Leikin, R. (2012). The role of multiple solution tasks in developing knowledge and creativity in geometry. *Journal of Mathematical Behavior*, 31, 73-90. doi:10.1016/j.jmathb.2011.11.001

Polya, G. (1945). *How to solve it*. Princeton: Princeton University Press.