Engaging with the Unexpected – Seek First to Understand

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INTRODUCTION

One of the many joys of teaching is seeing how different pupils tackle problems in different, and often unexpected, ways. In formal assessments this of course makes the marking process a little more complex as one needs to be vigilant about properly engaging with unexpected responses – particularly if the solution provided by the pupil is different to the marking guidelines. When pupils approach questions in unexpected ways we have a duty to be open-minded and to take the time necessary to properly explore and attempt to understand the reasoning behind the solution provided. In this article I share a recent episode that illustrates how engaging with the unexpected can lead to opportunities for deep learning.

EPISODE

During a Grade 12 lesson on Sequences and Series I wrote the following sum on the board and asked the class to express it using sigma notation:

$$(1 \times 2) + (5 \times 6) + (9 \times 10) + (13 \times 14) + \cdots + (81 \times 82)$$

The approach I was expecting pupils to adopt was to see the first term of each product as one linear sequence, and the second term of each product as a second linear sequence:

1st term of product: $1\,;5\,;9\,;13\,;\ldots;81 \quad (T_n = 4n - 3)$

2nd term of product: $2\,;6\,;10\,;14\,;\ldots;82 \quad (T_n = 4n - 2)$

Pupils would then have to give consideration to the number of terms in the sequence, for example by solving $81 = 4n - 3$. Having established that the sequences contains 21 terms, I was then expecting pupils to express the sum as:

$$\sum_{n=1}^{21}(4n-3)(4n-2)$$

As expected, this was indeed the approach followed by most of the class. However, one pupil provided the following response:

$$\sum_{i=1}^{21}\left(\sum_{n=1}^{4i-3}(2n)\right)$$

Before reading on, take a few minutes to engage with this solution and think about whether you would have given it serious consideration if the solution had been given by the pupil in a formal assessment such as a test or examination.

The solution took me completely by surprise, and my immediate reaction was to dismiss it. However, given that the pupil was on hand I was able to ask him to explain the reasoning behind his solution. He began by remarking that the sum of the first n positive integers can be expressed as:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

From this it follows that:

$$n(n+1) = 2\sum_{i=1}^{n} i$$

Returning to the original sequence, $(1 \times 2) + (5 \times 6) + (9 \times 10) + (13 \times 14) + \cdots + (81 \times 82)$, the pupil then explained that each product could be seen as double the sum of the first i positive integers, i being the first number of each product. So, for example, the second product (5×6) is twice the sum of the first 5 positive integers, and the last product (81×82) is twice the sum of the first 81 positive integers. The pupil had thus reconceptualised the problem as a summation of 21 separate sums:

$$2(1) + 2(1+2+\cdots+5) + 2(1+2+\cdots+9) + \cdots + 2(1+2+\cdots+81)$$

$$= 2\sum_{n=1}^{1} n + 2\sum_{n=1}^{5} n + 2\sum_{n=1}^{9} n + \cdots + 2\sum_{n=1}^{81} n$$

$$= \sum_{i=1}^{21}\left(2\sum_{n=1}^{4i-3} n\right)$$

$$= \sum_{i=1}^{21}\left(\sum_{n=1}^{4i-3} (2n)\right)$$

Although there is still something about the notation that leaves me a little uneasy, the reconceptualization of the sequence is rather brilliant, and the pupil's explanation behind his solution led to some wonderful discussion and deep learning.

CONCLUDING COMMENTS

What I hope I have illustrated in this short article is that proper engagement with the unexpected can often lead to deep learning experiences – for both the pupil as well as the teacher. Particularly when marking formal assessments, we need to constantly remind ourselves to be careful of dismissing a solution out of hand simply because it looks different to anything we might have been expecting. Our constant and vigilant byword should be 'seek first to understand', a good moral both mathematically speaking and otherwise.