Generalising the Number of Polygon Diagonals

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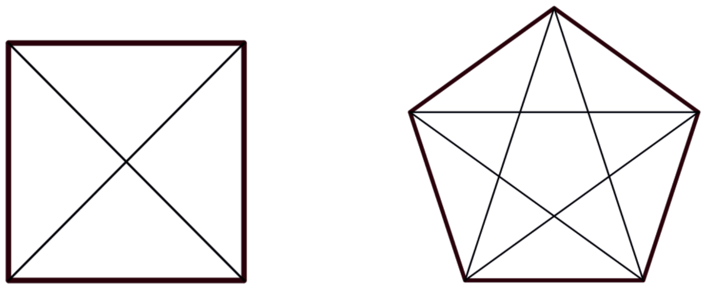
INTRODUCTION

As part of a Grade 11 investigation on quadratic patterns I included the following activity on determining the number of diagonals in regular polygons:

A square has 2 diagonals while a pentagon has 5 diagonals. This is illustrated below:



a) How many diagonals are there in a hexagon?

b) How many diagonals are there in a dodecagon, a 12-sided polygon?

c) Determine a formula for the number of diagonals in an n-sided polygon.

The reader is encouraged to engage with the activity before reading on.

GENERALISATION STRATEGIES

The first part of the activity, determining the number of diagonals in a hexagon, was given to encourage pupils to engage physically with the mathematical context by drawing a hexagon, adding in the diagonals, and then counting them in a systematic way. As one would expect, pupils managed this process with relative ease. The second part of the activity, determining the number of diagonals in a dodecagon, was given so as to provide a specific numerical case that would be too cumbersome to determine by physical counting. Although one or two pupils still attempted to draw a 12-sided polygon and count the diagonals, they soon abandoned this as being an impractical approach. Many pupils noted that since a square has 2 diagonals, a pentagon has 5 diagonals and a hexagon has 9 diagonals, we have $2\,;5\,;9$ as the first three terms of a sequence. Given that the investigation related to quadratic patterns, most pupils simply assumed (correctly) that the sequence was quadratic, i.e. that the second difference was constant, and used a standard method to find the general formula for the k^{th} term of the sequence: $T_k = \frac{1}{2}k^2 + \frac{3}{2}k$.

The problem with this approach is that the sequence starts with a square, i.e. a 4-sided figure, and as such the k^{th} term of the sequence actually applies to a polygon with $k+3$ sides. Pupils who realised this were able to correctly calculate the number of diagonals in a 12-sided polygon, namely 54, by determining the value of T_9. Pupils who were unable to see this critical connection erroneously calculated the number of diagonals as $T_{12} = 90$.

The final part of the activity was to establish a general formula for the number of diagonals in an n-sided polygon. A number of pupils who had already established the general formula for the sequence $2\,;5\,;9$, and who realised that the sequence was out of sync with what was required, were able to adjust the formula by replacing k with $n-3$ to arrive correctly at the formula $T_n = \frac{1}{2}(n-3)^2 + \frac{3}{2}(n-3)$. An alternative approach would have been to extend the sequence back by three terms to give $-1\,;\,-1\,;\,0\,;\,2\,;5\,;9$ and then determine a general formula for it directly, realising of course that the formula is restricted to $n \geq 3$.

While most pupils attempted to arrive at a general expression (the third part of the activity) before determining the number of diagonals in a dodecagon (the second part of the activity), this was not the intention. As previously mentioned, the purpose of asking pupils to determine the number of diagonals in a dodecagon was to provide a specific numerical case where direct drawing and counting would be impractical. The intention was thus to encourage the development of a more general approach to the problem through engagement with a specific instance, and for pupils to make use of the specific as a *generic example* en route to the general. Some pupils were able to do this very elegantly by reasoning along the following lines:

> A dodecagon has 12 sides and hence twelve vertices. Each vertex has 9 diagonals connected to it. Multiplying 12 by 9 gives 108. However, this in effect counts every diagonal twice (since each diagonal is connected to two vertices) so we need to divide this total by 2 to give a final answer of 54. Extending this logic to an n-sided polygon, each of the n vertices has $n-3$ diagonals connected to it (since a given vertex cannot be connected by a diagonal to either itself or the two vertices on either side of it). Multiplying n by $n-3$ and then dividing by 2 to correct for the double counting gives our final formula: $T_n = \frac{n(n-3)}{2}$.

While far more elegant than the previously established formula $T_n = \frac{1}{2}(n-3)^2 + \frac{3}{2}(n-3)$, the two formulas are nonetheless algebraically equivalent and represent two different approaches to solving the task.

GENERALISATION WITHOUT ALGEBRA

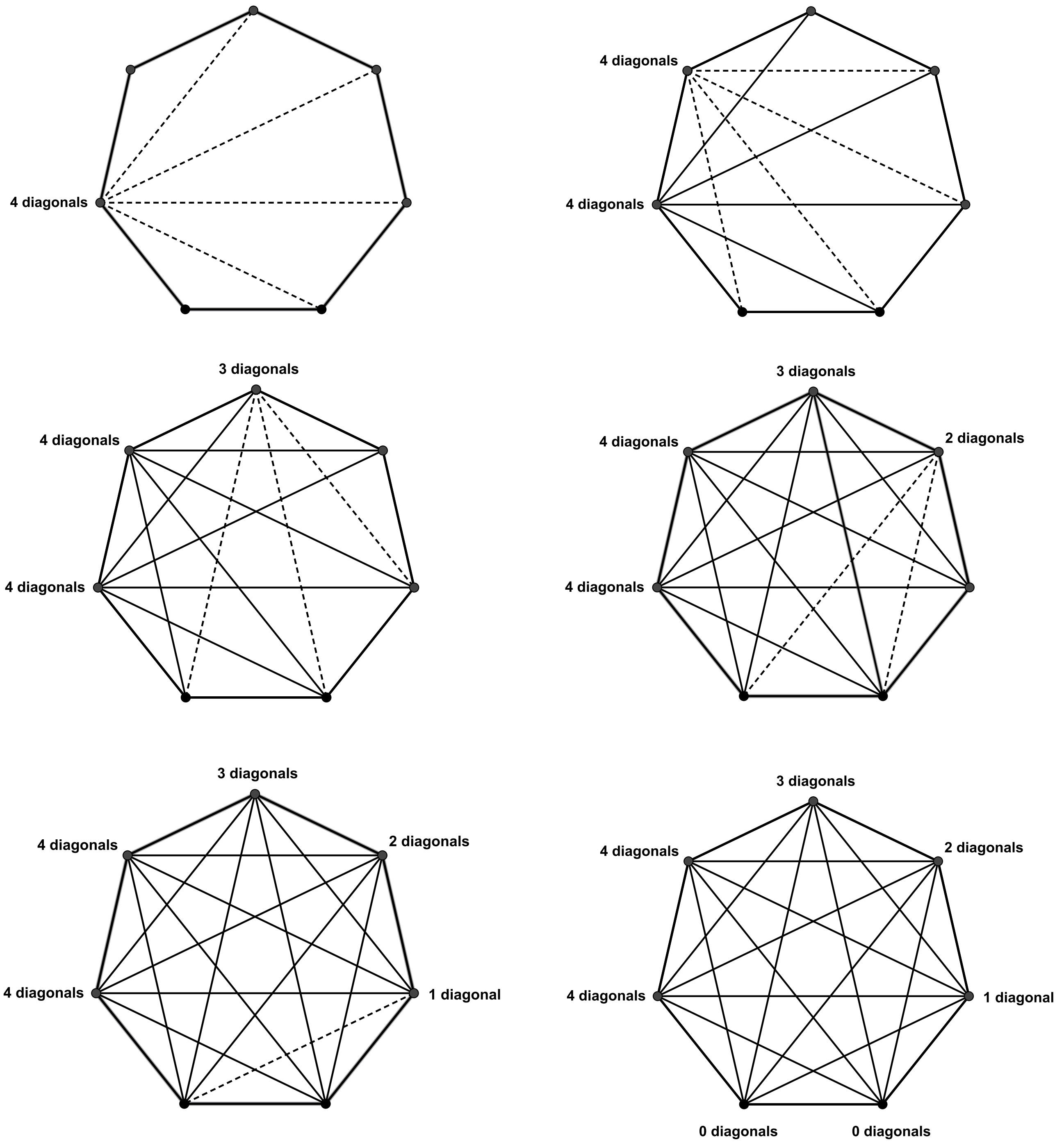
A colleague[4] and I used a slightly modified version of this activity with Grade 6 and 7 pupils. The scenario itself was presented in exactly the same way, but the questions then required pupils to determine the number of diagonals in various specific polygons, namely (a) a 6-sided polygon, (b) a 7-sided polygon, (c) an 8-sided polygon, (d) a 12-sided polygon, (e) a 20-sided polygon, and finally (f) a 100-sided polygon. In addition, diagrams of a regular hexagon, heptagon and octagon were provided.

Although algebraic symbolism allows for compact and semantically unambiguous general statements, expressions of generality are not necessarily restricted to the language of algebra. An important precursor to algebraic generalisation is the ability to articulate general patterns or structures using natural language. It is this aspect of the generalisation process that this activity was meant to foreground. As such, allowing pupils to work in pairs was particularly effective as it encouraged the verbal articulation of inner reasoning as pupils shared their ideas.

[4] Simon Kroon, some of whose shared experiences I have incorporated into this article.

Parts (a), (b) and (c), i.e. determining the number of diagonals in a hexagon, heptagon and octagon, were intended to engage pupils in the physical process of drawing in the diagonals on the diagrams provided, and then counting them. Although most pupils began this process in an unstructured and rather haphazard way, the need for a more systematic approach to drawing and counting the diagonals was quickly realised. Some pupils attempted to use different coloured pencils to keep track of different diagonals, while others used the inherent symmetry of the polygons as a means of checking to see whether all possible diagonals had been accounted for – for example by comparing the left-hand side of the diagram with the right-hand side.

Some pupils used the following systematic strategy, illustrated below for a heptagon:



The strategy involved beginning with a single vertex, drawing in all the diagonals that extended from it, and then systematically moving in a clockwise direction and repeating the process with each of the following vertices. In the case of a heptagon, while the first two vertices each required 4 diagonals to be drawn in, the third vertex required 3 additional diagonals, the fourth only 2 additional diagonals, and the fifth vertex only 1 additional diagonal. The sixth and seventh vertices required no further diagonals to be drawn in. The total number of diagonals in a heptagon was then calculated as $4 + 4 + 3 + 2 + 1 = 14$. Pupils were then able to confirm that a similar pattern arose in the case of the diagonals of an octagon, viz. $5 + 5 + 4 + 3 + 2 + 1 = 20$. It was then a reasonably simple matter to extend the generalisation to other polygons. However, two important considerations arose. Firstly, note that the first two vertices of a 7-sided polygon each require 4 diagonals, and the first two vertices of an 8-sided polygon each require 5 diagonals. From this one can generalize that for a given polygon the first two vertices each require 3 fewer diagonals than there are sides or vertices. I was impressed that many pupils were able to provide an explanation for this, namely that a given vertex cannot be connected by a diagonal to either itself or the two vertices on either side of it. The second consideration relates to determining the sum itself. If we consider a 20-sided polygon, then the number of diagonals would be $17 + 17 + 16 + 15 + 14 + \cdots + 3 + 2 + 1$. Determining this sum without a calculator is a little tedious, but somewhat fortuitously we had recently discussed the so-called Gauss method of summing linear sequences. We were thus able to proceed as follows:

$$\begin{aligned} &17 + [17 + 16 + 15 + 14 + \cdots + 3 + 2 + 1] \\ &= 17 + \left[\frac{18 \times 17}{2}\right] \\ &= 17 + 9 \times 17 \\ &= 10 \times 17 \\ &= 170 \end{aligned}$$

REFLECTION & CONCLUDING COMMENTS

The pedagogical power of tasks such as the one illustrated in this article lies in the pictorial context in which the activity is embedded. Rather than a mere sequence of numbers, engagement with a pictorial context has the potential to open up a much greater variety of approaches. Furthermore, engagement with the context itself allows for the potential development of a far deeper sense of generality, and it is this aspect of the generalisation process that is so important. It is sad that valuable opportunities for genuine mathematical exploration so often get reduced to a mechanical process of determining an algebraic expression for the general term of a numerical sequence extracted from the context. Such a mechanistic course, even if carried out efficiently, divorces the process from the original context and as such limits the potential insight that can be gleaned from the context itself. The unfortunate result of this is that the focus shifts from the development of a genuine sense of generality to the mere establishment of an algebraic relationship.

A pupil who has meaningfully engaged with the original context and is able to articulate the number of diagonals in a 20-sided polygon as $17 + 17 + 16 + 15 + 14 + \cdots + 3 + 2 + 1$, although not yet able to express this generality using algebraic symbolism, has nonetheless developed a far more profound sense of the general than a pupil who arrives at the formula $T_n = \frac{1}{2}(n-3)^2 + \frac{3}{2}(n-3)$ through nothing more than a mechanistic treatment of an extracted numerical sequence. Both processes of course have value and importance, but it is critical that we constantly reflect on what we hope pupils will extract from a particular activity, and on how best we might bring this about.