A Multiple Solution Problem

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INTRODUCTION

Learners often tend to compartmentalize the different sections of the school Mathematics syllabus. One way to break down these perceived boundaries is to engage with problems that can lead to multiple and varied solutions. Exploration of such multiple solution problems encourages reflection and flexibility of thought, thereby enhancing one's repertoire of problem solving skills and approaches for future challenges (De Villiers, 2016; Polya, 1945).

In this article I explore a variety of solutions to a problem that appeared in the Senior Individual paper of the 2015 South African Team Competition.

THE PROBLEM

The problem was presented without a diagram and was posed as follows:

> In ΔABC, D is the midpoint of BC. If $\angle$ADB = 45° and $\angle$ACD = 30°, determine $\angle$DAB.

Since the question was posed without a diagram it makes good sense to draw a diagram that captures the described scenario. A natural reaction would probably be to draw an *acute* angled triangle ABC and proceed from there. However, we would do well to question the justification of such an assumption. In fact, it turns out that $\angle$B is actually obtuse, as the following explanation shows. Draw line segment CB having length 2, with midpoint D. From D, construct a ray tilted at 45° to CB. Also draw the perpendicular at B as shown in Figure 1.

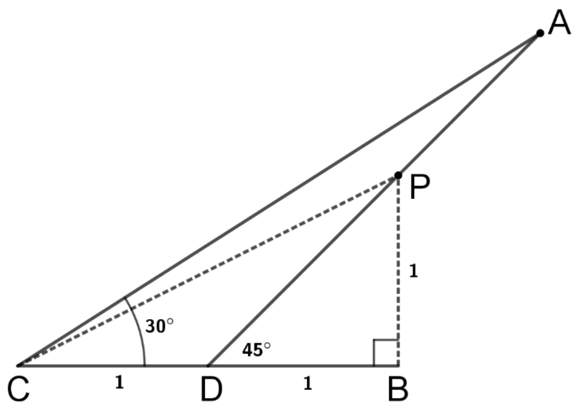

FIGURE 1: Capturing the problem diagrammatically

If the perpendicular from B intersects DA at P, we have $\tan(P\hat{C}B) = \frac{1}{2}$ since PB = DB = 1. Now draw a ray at C tilted at 30°, as described in the problem. Since $\tan 30° = \frac{1}{\sqrt{3}} > \frac{1}{\sqrt{4}} = \frac{1}{2}$, CA intersects the perpendicular from B at a point higher than P. We thus have DA > DP, making $\angle$ABC obtuse.

A BRIEF NOTE ABOUT UNITS

In our daily lives we make frequent references to measurements, and these are made in relation to a known fixed unit of measurement, e.g. metres, centimetres, feet etc. In mathematics however, we often do not specify the unit of measurement. For example, when we say that a line segment AB = 15, we can well ask 15 what? The answer is that we have already decided on some unspecified unit of measurement, and it need not be a standard unit such as a centimetre or metre. All we are saying is that AB is 15 times as long as that unspecified unit. We do this all the time. For example, when we draw a system of axes we place 0 and 1 randomly on the horizontal axis. What we are in fact doing is specifying the unit of measurement as the distance between the points 0 and 1. This means that if in some problem no lengths are specified, as in the problem posed in this article, we are free to decide on our unit length, usually the length of some line segment in our figure. We are just as free not to specify our chosen unit. Thus we can say, for example, "let a, b and c be the sides of a triangle" or "let 1, b and c be the sides of a triangle" in a problem where the unit is unspecified. The decision to let CB conveniently have length 2 in Figure 1 is thus entirely justified.

1. A TRIGONOMETRIC SOLUTION

Trigonometry offers a number of possible solutions to the problem. The first trigonometric solution is perhaps the simplest and most direct. If we let CB = 2 then CD = DB = 1.

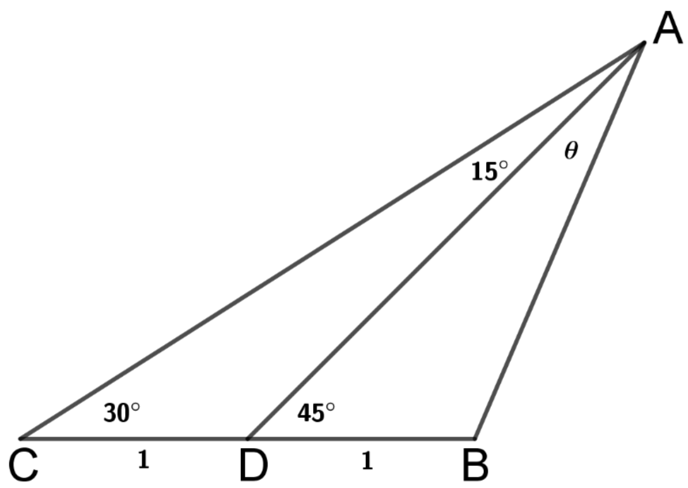

FIGURE 2: A trigonometric solution

Letting $\angle$DAB $= \theta$ and applying the sine rule to ΔABC gives $\frac{2}{\sin(15°+\theta)} = \frac{\text{AB}}{\sin 30°}$ from which we have $\frac{2}{\sin(15°+\theta)} = 2AB$. If we now apply the sine rule to ΔABD then $\frac{\text{AB}}{\sin 45°} = \frac{1}{\sin\theta}$ from which it follows that $\text{AB} = \frac{\sin 45°}{\sin\theta} = \frac{\sqrt{2}}{2\sin\theta}$ and hence $\frac{2}{\sin(15°+\theta)} = \frac{\sqrt{2}}{\sin\theta}$. We thus have $2\sin\theta = \sqrt{2}\sin(15° + \theta)$ from which $\theta = 30°$ by inspection.

2. USING TRIGONOMETRY AND MANIPULATION

The above solution, although elegant and concise, is not necessarily easy to spot. In the following solution we solve for θ by isolating it through a process of manipulation.

As before, let CD = DB = 1 and $\angle$DAB = θ.

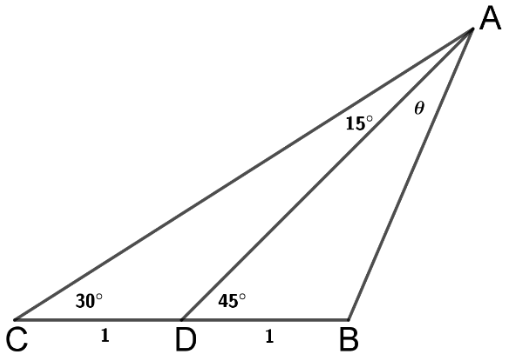

FIGURE 3: Using trigonometry and manipulation

We begin by applying the sine rule to ΔACD and ΔADB:

$$\frac{1}{\sin 15°} = \frac{\text{AD}}{\sin 30°} \quad \& \quad \frac{1}{\sin\theta} = \frac{\text{AD}}{\sin(180° - (45° + \theta))} \quad \to \quad \frac{1}{\sin\theta} = \frac{\text{AD}}{\sin(45° + \theta)}$$

We thus have $\frac{1}{\text{AD}} = \frac{\sin 15°}{\sin 30°}$ and $\frac{1}{\text{AD}} = \frac{\sin\theta}{\sin(45°+\theta)}$ from which we can write:

$$\sin 15° \sin(45° + \theta) = \sin 30° \sin\theta$$

Expanding $\sin(45° + \theta)$ using the compound angle formula gives:

$$\sin 15° \,(\sin 45° \cos\theta + \cos 45° \sin\theta) = \frac{1}{2}\sin\theta$$

$$\therefore \quad \sin 15° \frac{\sqrt{2}}{2}(\cos\theta + \sin\theta) = \frac{1}{2}\sin\theta$$

$$\therefore \quad \sin 15° \,(\cos\theta + \sin\theta) = \frac{\sqrt{2}}{2}\sin\theta$$

We thus have $\frac{\cos\theta + \sin\theta}{\sin\theta} = \frac{\sqrt{2}}{2\sin 15°}$. If we re-write $\sin 15°$ as $\sin(45° - 30°)$, expand it using the compound angle formula and determine the special angle ratios, we can then write:

$$\frac{\cos\theta + \sin\theta}{\sin\theta} = \frac{2}{\sqrt{3} - 1}$$

$$\therefore \quad \frac{\cos\theta}{\sin\theta} + 1 = \frac{2}{\sqrt{3} - 1}$$

$$\therefore \quad \frac{\cos\theta}{\sin\theta} = \frac{2}{\sqrt{3} - 1} - 1$$

We can manipulate $\frac{2}{\sqrt{3}-1} - 1$ as follows:

$$\frac{2}{\sqrt{3} - 1} - 1 = \frac{2 - (\sqrt{3} - 1)}{\sqrt{3} - 1} = \frac{3 - \sqrt{3}}{\sqrt{3} - 1} = \frac{\sqrt{3}(\sqrt{3} - 1)}{\sqrt{3} - 1} = \sqrt{3}$$

We thus have $\frac{\cos\theta}{\sin\theta} = \sqrt{3}$ from which it follows that $\tan\theta = \frac{1}{\sqrt{3}}$ and thus $\theta = 30°$.

There is a different solution using trigonometry and manipulation that first solves for AD by applying the sine rule to triangle ACD, using the cosine rule in triangle ADB to show that AB = $\sqrt{2}$, and then finally applying the sine rule to triangle ADB to obtain the desired solution. The details are left to the reader to explore.

3. A SOLUTION THAT BYPASSES THE SINE AND COSINE RULES

Although the compound angle formulae for sine and cosine still form part of the South African school Mathematics syllabus, the compound angle formulae for tan no longer do.

$$\tan(\mathrm{A} \pm \mathrm{B}) = \frac{\tan \mathrm{A} \pm \tan \mathrm{B}}{1 \mp \tan \mathrm{A} \tan \mathrm{B}}$$

We will make use of the tan compound angle expansion in the solution that follows. As previously, let CD and DB have length 1 and let $\angle$DAB = θ.

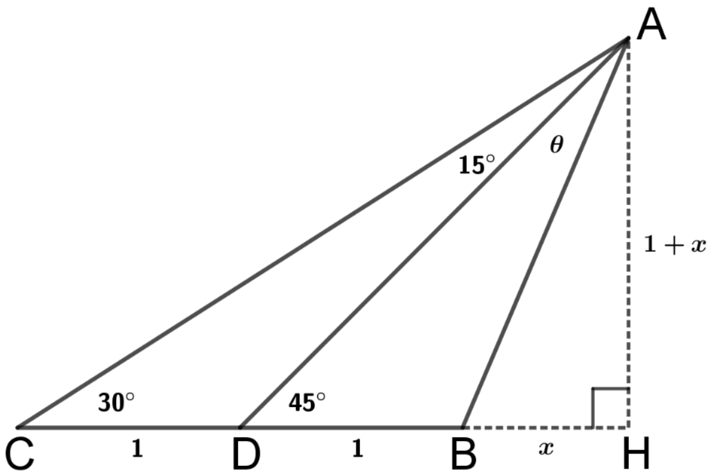

FIGURE 4: Bypassing the sine and cosine rules

AH is now drawn perpendicular to CB with H on CB produced. Let BH = x. Since ADH is a right-angled isosceles triangle we have AH = DH = $1 + x$.

Since $\angle$CAH = 60° we have $\tan \mathrm{C\hat{A}H} = \tan 60°$ and thus $\frac{1+1+x}{1+x} = \sqrt{3}$. This can be written in the form $\frac{1}{1+x} + 1 = \sqrt{3}$ which, on rearrangement, gives $1 + x = \frac{1}{\sqrt{3}-1} = \frac{1+\sqrt{3}}{2}$, from which $x = \frac{\sqrt{3}-1}{2}$. Now, in triangle BAH we have $\tan \mathrm{B\hat{A}H} = \frac{x}{1+x}$. However, since $\angle$BAH = $45° - \theta$, from the compound angle formula for tan we have $\tan(45° - \theta) = \frac{\tan 45° - \tan\theta}{1 + \tan 45° \tan\theta} = \frac{1-\tan\theta}{1+\tan\theta}$. Equating these two expressions for $\tan \mathrm{B\hat{A}H}$ gives $\frac{1-\tan\theta}{1+\tan\theta} = \frac{x}{1+x}$. And since $x = \frac{\sqrt{3}-1}{2}$ and $1 + x = \frac{1+\sqrt{3}}{2}$, we have:

$$\frac{1 - \tan\theta}{1 + \tan\theta} = \frac{\frac{\sqrt{3}-1}{2}}{\frac{1+\sqrt{3}}{2}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$$

By inspection it thus follows that $\tan\theta = \frac{1}{\sqrt{3}}$ and hence that $\theta = 30°$.

4. USING ANALYTICAL GEOMETRY AND ALGEBRA

Analytical geometry is a powerful tool for solving problems. In the following solution the diagram is positioned on a set of axes with the origin at D. B and C consequently have coordinates $(1;0)$ and $(-1;0)$ respectively. Since $\angle$ ADB = 45°, this means that line segment AD has equation $y = x$. Since A lies on AD its x-coordinate and y-coordinate will be the same. Let A have coordinates $(a;a)$ and let AB make an angle of α with the positive x-axis.

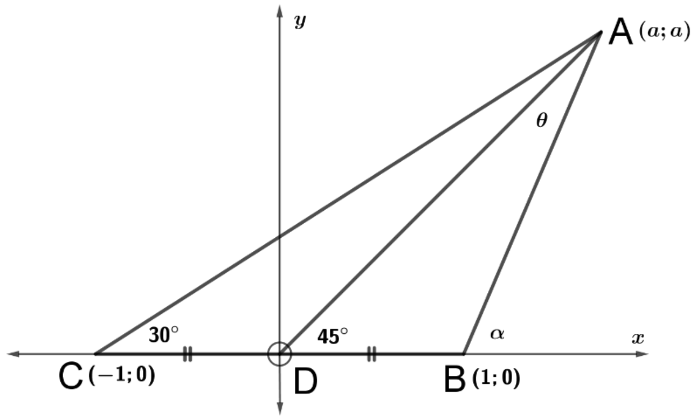

FIGURE 5: Positioning the diagram in the Cartesian plane

Now, since the gradient of line segment AC is $\tan 30° = \frac{1}{\sqrt{3}}$, we have $\frac{a-0}{a+1} = \frac{1}{\sqrt{3}}$. This can be rearranged to show that $a = \frac{1+\sqrt{3}}{2}$. Since the gradient of line segment AB is $\frac{a}{a-1}$ we have $\tan\alpha = \frac{a}{a-1}$. But α is the exterior angle of triangle ADB, thus $\theta = \alpha - 45°$ and hence $\tan\theta = \tan(\alpha - 45°)$. Using the compound angle expansion for tan gives:

$$\tan\theta = \tan(\alpha - 45°) = \frac{\tan\alpha - \tan 45°}{1 + \tan\alpha \tan 45°}$$

$$= \frac{\frac{a}{a-1} - 1}{1 + \frac{a}{a-1}} = \frac{1}{2a-1} = \frac{1}{2\left(\frac{1+\sqrt{3}}{2}\right) - 1} = \frac{1}{\sqrt{3}}$$

Thus, as before, we have $\tan\theta = \frac{1}{\sqrt{3}}$ from which it follows that $\theta = 30°$.

5. USING ALGEBRA AND SIMILARITY

Use the diagram as in solution 3, but let AH = 1. This means that DH = 1 and hence CD = DB = $1 - x$ as shown in Figure 6. Note that in triangle ACH we have $\sin 30° = \frac{1}{\text{AC}}$ from which it follows that AC = 2. Using the theorem of Pythagoras in triangle ACH we have $(1 - x + 1 - x + x)^2 + 1^2 = 2^2$ which can be simplified to $(2 - x)^2 = 3$. Multiplying this out and rearranging we can show that $1 + x^2 = 4x$. In addition, note that $(1 - x)^2$ simplifies to $1 - 2x + x^2$, and since we have previously shown that $1 + x^2 = 4x$ this means that $(1 - x)^2 = 2x$ and thus $1 - x = \sqrt{2x}$. We thus have two important results:

$$1 + x^2 = 4x \qquad 1 - x = \sqrt{2x}$$

If we now consider triangles ABD and CBA:

$$\frac{\text{AB}}{\text{CB}} = \frac{\sqrt{1+x^2}}{2(1-x)} = \frac{\sqrt{4x}}{2\sqrt{2x}} = \frac{1}{\sqrt{2}}$$

$$\frac{\text{AD}}{\text{CA}} = \frac{\sqrt{1^2+1^2}}{2} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\frac{\text{BD}}{\text{BA}} = \frac{1-x}{\sqrt{1+x^2}} = \frac{\sqrt{2x}}{\sqrt{4x}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

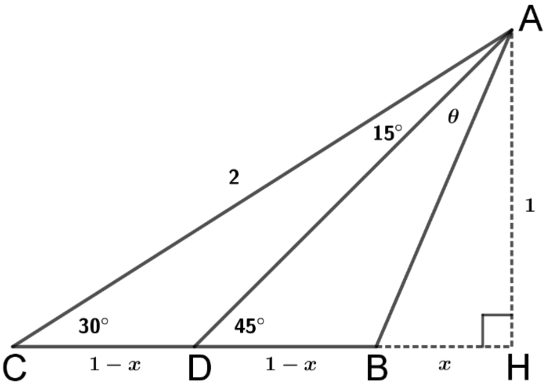

FIGURE 6: Using algebra and similarity

Hence triangles ABD and CBA are similar,
from which it follows that ∠BAD = ∠BCA = 30°.

6. A GEOMETRIC SOLUTION

While solutions that employ the use of trigonometry, algebra and analytical geometry are always acceptable, a purely geometric solution to a geometrical problem is not only desirable but often elegant and intellectually fulfilling. I have therefore left this solution to the end.

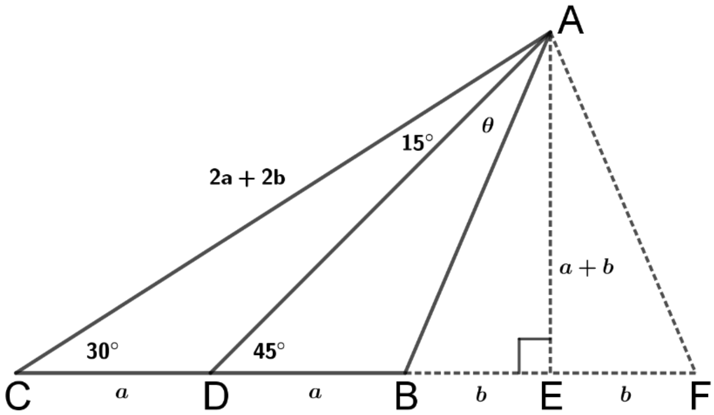

FIGURE 7: A purely geometric solution

Let CD = DB = a. Draw AE perpendicular to CB with E on CB produced. Let BE = b. Produce BE to F such that EF = b. Join A to F. Since ∠ADE = ∠DAE = 45°, triangle AED is isosceles and hence AE = DE = $a + b$. Since $\sin 30° = \frac{1}{2}$, AC = 2AE = $2a + 2b$. But CF = $2a + 2b$. It thus follows that triangle ACF is isosceles, and ∠F = $\frac{1}{2}(180° - 30°) = 75°$ and hence ∠BAE = ∠FAE = 15°. Rather neatly we thus have ∠DAB = $45° - 15° = 30°$.

REFERENCES

De Villiers, M. (2016). A Multiple Solution Task: A South African Mathematics Olympiad Problem. *Learning and Teaching Mathemati*cs, 20, 18-20.

Polya, G. (1945). *How to solve it*. Princeton: Princeton University Press.