Euclidean Geometry – Nurturing Multiple Solutions

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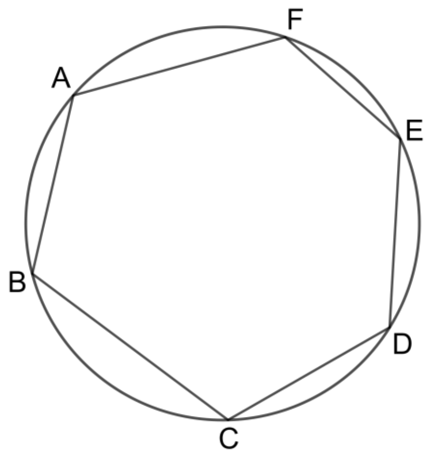
INTRODUCTION

In a previous article (Samson, 2015) I highlighted the importance of providing pupils with opportunities to engage with geometric contexts in explorative and flexible ways. Such geometric scenarios should be posed with minimal directional guidance, thereby encouraging creative mathematical thinking. The beauty of such contexts is that they allow pupils to tackle the question in different ways. As the final question in our November 2017 Grade 11 Paper 2 examination we included the following:

You are given the cyclic hexagon below. Prove, with reasons, that $\hat{A} + \hat{C} + \hat{E} = 360°$.

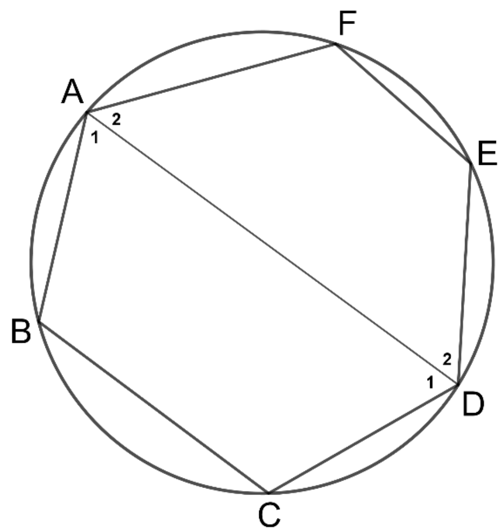


This was one of the questions I was assigned to mark, and I was delighted with the variety of approaches our pupils managed to come up with. Five different solutions follow:

SOLUTION 1

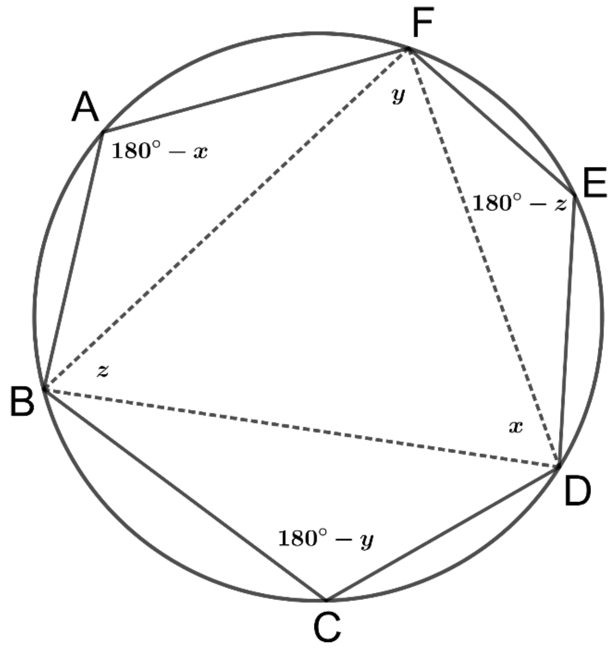
Draw in line AD, thereby creating cyclic quadrilaterals ABCD and ADEF. Since the opposite angles of a cyclic quadrilateral are supplementary, we have $\hat{A}_1 + \hat{C} = 180°$ and $\hat{A}_2 + \hat{E} = 180°$.
Combining these gives $\hat{A}_1 + \hat{A}_2 + \hat{C} + \hat{E} = 2 \times 180°$, hence $\hat{A} + \hat{C} + \hat{E} = 360°$.

This was the anticipated solution, and many pupils approached the question in this way.



SOLUTION 2

Rather than creating two separate cyclic quadrilaterals by drawing in a diagonal, many pupils created three overlapping cyclic quadrilaterals as illustrated below:



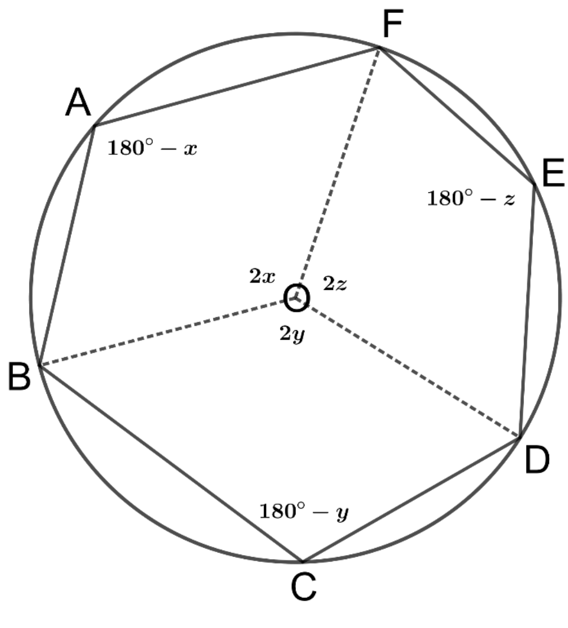
Letting angles $B\hat{D}F$, $B\hat{F}D$ and $D\hat{B}F$ be respectively x, y and z, then angles $\hat{A}$, $\hat{C}$ and $\hat{E}$ are respectively $180° - x$, $180° - y$ and $180° - z$. We thus have:

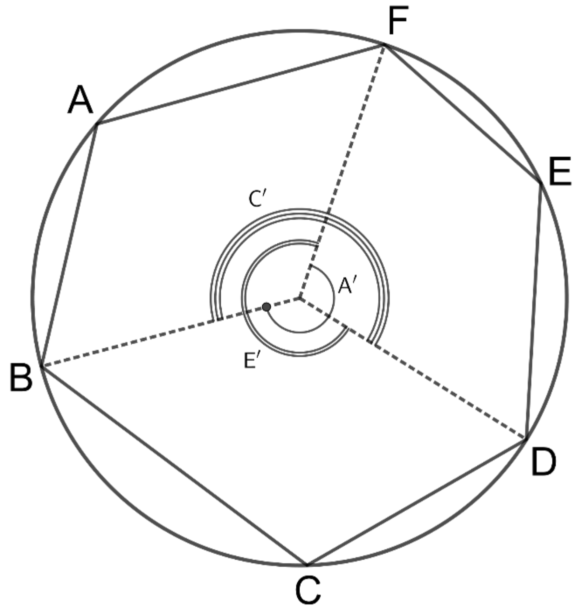
$$\begin{aligned} \hat{A} + \hat{C} + \hat{E} &= 180° - x + 180° - y + 180° - z \\ &= 540° - (x + y + z) \\ &= 360° \end{aligned}$$

SOLUTION 3

In this solution, vertices B, D and F are joined to the centre of the circle, O. If we let $F\hat{O}B = 2x$ then reflex $F\hat{O}B = 360° - 2x$ and hence $\hat{A} = 180° - x$ (angle at centre is twice the angle at the circumference). Similarly, letting $B\hat{O}D = 2y$ and $F\hat{O}D = 2z$ we have $\hat{C} = 180° - y$ and $\hat{E} = 180° - z$. We thus have $\hat{A} + \hat{C} + \hat{E} = 180° - x + 180° - y + 180° - z = 540° - (x + y + z)$.

But $2x + 2y + 2z = 360°$, thus $x + y + z = 180°$, from which it follows that $\hat{A} + \hat{C} + \hat{E} = 360°$.

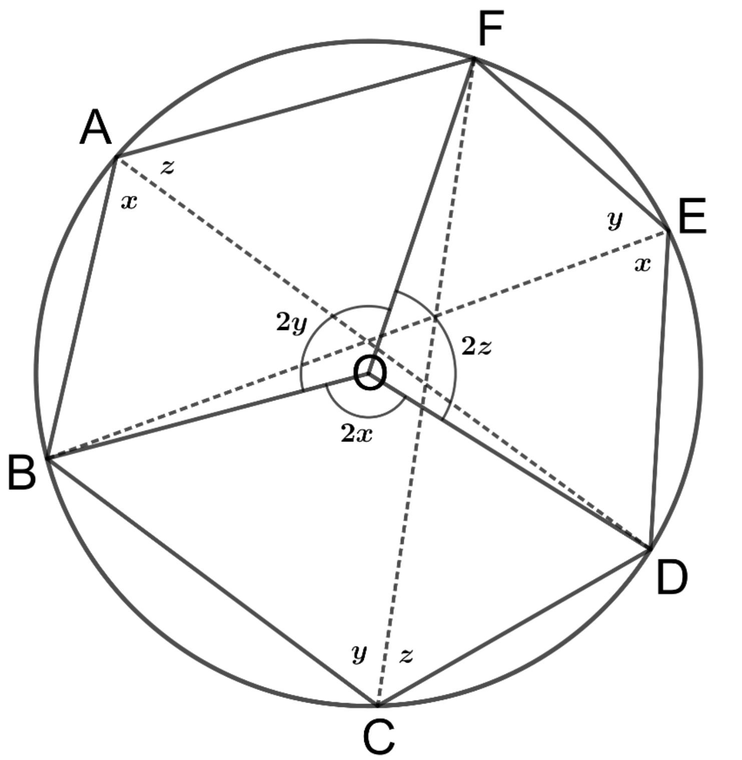




One pupil used the essence of solution 3, but with beautiful visual reasoning. Since the angle at the centre is twice the angle at the circumference, we have $\hat{A}' = 2\hat{A}$, $E' = 2\hat{E}$ and $\hat{C}' = 2\hat{C}$. Starting from the dot on $\hat{A}'$ and moving in an anticlockwise direction, $\hat{A}' + \hat{E}' + \hat{C}'$ forms a circular path around the centre of the circle for two full revolutions. We thus have $\hat{A}' + \hat{E}' + \hat{C}' = 720°$, i.e. $2\hat{A} + 2\hat{E} + 2\hat{C} = 720°$, from which it follows that $\hat{A} + \hat{E} + \hat{C} = 360°$.

SOLUTION 4

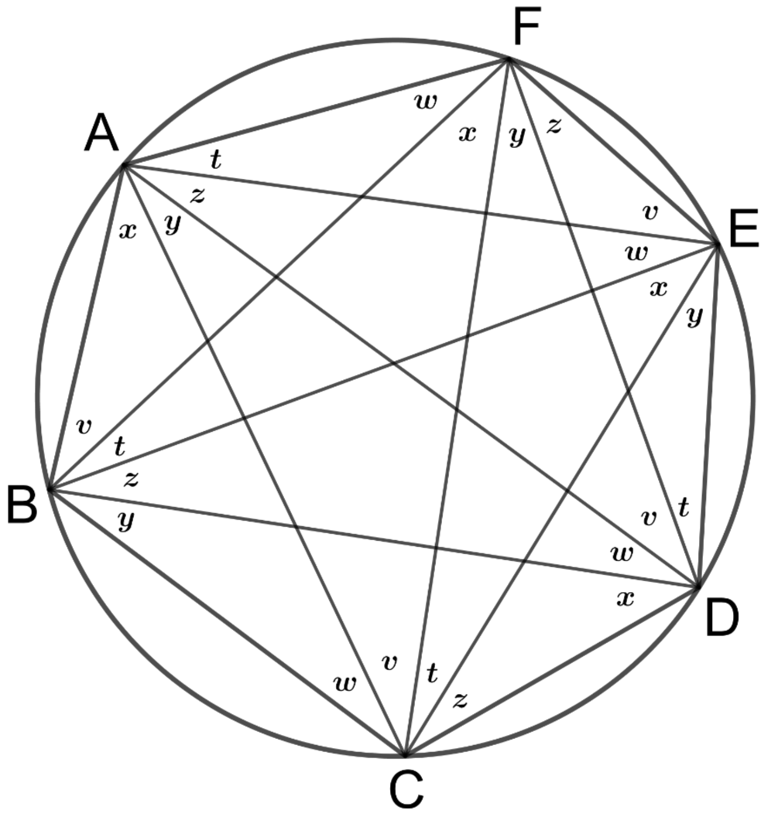
In this solution, vertices B, D and F are once again joined to the centre of the circle, O. However, instead of using reflex angles one pupil used the approach shown below.



If we let $B\hat{O}D = 2x$ then $B\hat{A}D = B\hat{E}D = x$. Similarly, letting $F\hat{O}B = 2y$ and $F\hat{O}D = 2z$ we have respectively $F\hat{E}B = F\hat{C}B = y$ and $F\hat{C}D = F\hat{A}D = z$. Since $\hat{A} = x + z$, $\hat{E} = x + y$ and $\hat{C} = y + z$, we have $\hat{A} + \hat{C} + \hat{E} = 2x + 2y + 2z$. But $2x + 2y + 2z = 360°$ (revolution), thus $\hat{A} + \hat{C} + \hat{E} = 360°$.

SOLUTION 5

Although no pupil used this approach, one could also draw in the three diagonals from each vertex of the hexagon and then make use of the fact that a chord subtends equal angles in the same segment.



Note that $\hat{A}+\hat{C}+\hat{E} = (x+y+z+t)+(w+v+t+z)+(v+w+x+y)$
$= 2(t+v+w+x+y+z)$

Close inspection shows that the interior angle sum of each of the eight triangles formed from the vertices of the hexagon is exactly $t+v+w+x+y+z$. In other words $t+v+w+x+y+z=180°$, from which it follows that $\hat{A}+\hat{C}+\hat{E}=360°$.

CONCLUDING COMMENTS

The variety of solutions that pupils managed to come up with illustrates how a simple yet suitably open-ended geometrical context has the potential to promote creative and flexible engagement as well as stimulate higher order thinking. Given the space to think creatively, and with minimal directional guidance, pupils can indeed rise to the challenge.

REFERENCES

Samson, D. (2015). Devising explorative Euclidean geometry questions. *Learning and Teaching Mathematics, 19*, 13-16.