Finding the Height of a Triangle Given Two Angles and the Included Side

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This article presents a useful trigonometric formula that can be used to find the height of a triangle given the two base angles and the included side, i.e. the base length.

$$h = \frac{a.\tan B.\tan C}{\tan B + \tan C}$$

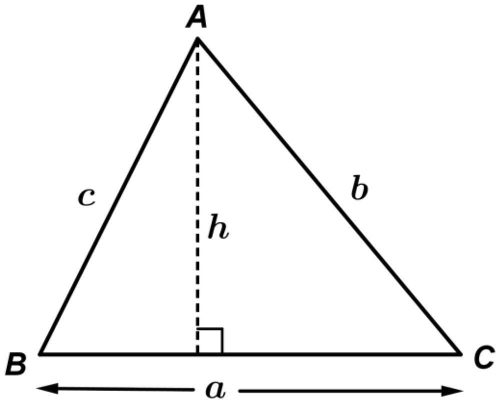

FIGURE 1: A useful trigonometric formula.

Let's begin by proving the formula for an acute-angled triangle as illustrated in Figure 2.

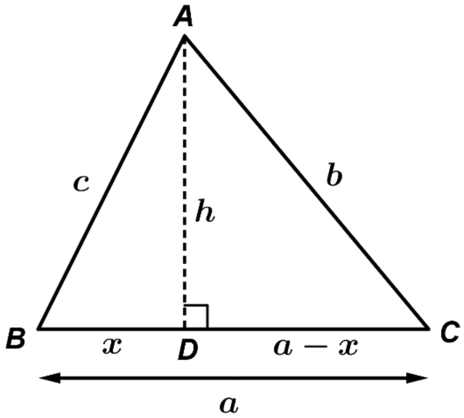

FIGURE 2: Proving the formula for an acute-angled triangle.

In ΔABD we have $\tan B = \frac{h}{x}$ from which it follows that: $x = \frac{h}{\tan B}$ . . . (1)

In ΔACD we have $\tan C = \frac{h}{a-x}$ from which it follows that: $a - x = \frac{h}{\tan C}$ . . . (2)

We can now substitute equation (1) into equation (2):

$$a - \frac{h}{\tan B} = \frac{h}{\tan C} \quad \therefore \quad a = \frac{h}{\tan B} + \frac{h}{\tan C}$$

Writing the right-hand side over a common denominator of $\tan B . \tan C$ we obtain:

$$a = \frac{h \tan C + h \tan B}{\tan B . \tan C}$$

$$\therefore \quad a = \frac{h(\tan C + \tan B)}{\tan B . \tan C}$$

Rearranging this gives the desired formula: $h = \dfrac{a . \tan B . \tan C}{\tan B + \tan C}$

Let us now consider the case of an obtuse-angled triangle. Does the formula still hold true? Consider the obtuse-angled triangle illustrated in Figure 3.

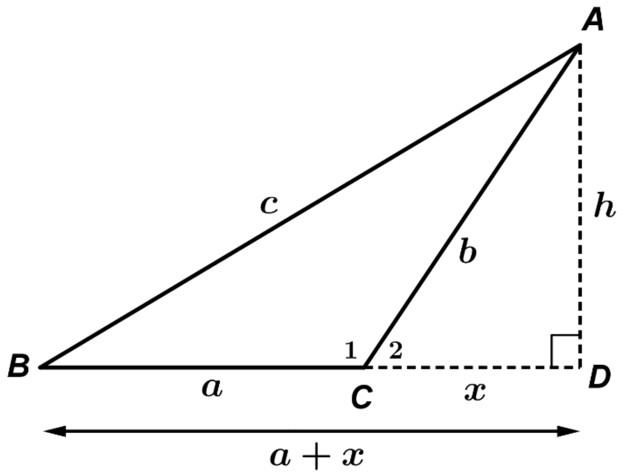

FIGURE 3: Proving the formula for an obtuse-angled triangle.

In ΔABD we have $\tan B = \dfrac{h}{a+x}$ from which it follows that: $a + x = \dfrac{h}{\tan B}$ . . . (1)

In ΔACD we have $\tan C_2 = \dfrac{h}{x}$. However, since $\hat{C}_2 = 180° - \hat{C}_1$ we have $\tan(180° - C_1) = \dfrac{h}{x}$, hence $-\tan C_1 = \dfrac{h}{x}$ and thus: $x = -\dfrac{h}{\tan C_1}$ . . . (2)

We can now substitute equation (2) into equation (1):

$$a - \frac{h}{\tan C_1} = \frac{h}{\tan B} \quad \therefore \quad a = \frac{h}{\tan B} + \frac{h}{\tan C_1}$$

Writing the right-hand side over a common denominator and rearranging as before gives the desired formula: $h = \dfrac{a . \tan B . \tan C}{\tan B + \tan C}$

Let us now apply the formula to a typical question:

Question:
Determine the perpendicular height of triangle ABC.

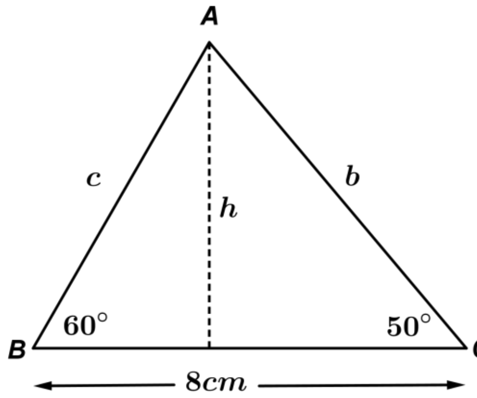

FIGURE 4: Applying the formula to a typical question.

Ordinarily with this type of question one would proceed by determining the size of angle A and then using the sine rule to determine either c or b. From this one could then use basic trigonometry in a right-angled triangle to determine the perpendicular height h.

Since $\hat{A} = 180° - (60° + 50°) = 70°$, using the sine rule in triangle ABC we have $\frac{c}{\sin 50°} = \frac{8}{\sin 70°}$, thus $c = \frac{8 \sin 50°}{\sin 70°} = 6{,}5216...$ Working within a right-angled triangle we now have $\sin 60° = \frac{h}{c}$ from which it follows that $h = c \sin 60° = 6{,}5216 \times \sin 60° = 5{,}648\ cm$ correct to three decimal places.

However, instead of using this multi-step process, we could simply use the formula we proved earlier to arrive directly at the answer in a single step:

$$h = \frac{8 \tan 60° \tan 50°}{\tan 60° + \tan 50°} = 5{,}648\ cm$$

As shown in the above example, when determining the perpendicular height of a triangle, given the two base angles and the included side (i.e. the base length), this useful alternative formula offers a quick and more direct route to the solution.