Which Picture and Why? Being Deliberate and Explicit when Using Representations

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A MULTITUDE OF REPRESENTATIONS

What is 12 × 7? The answer of course is 84. And that's the end of the story … or is it? I recently posed the question in Figure 1 to a group of teachers.

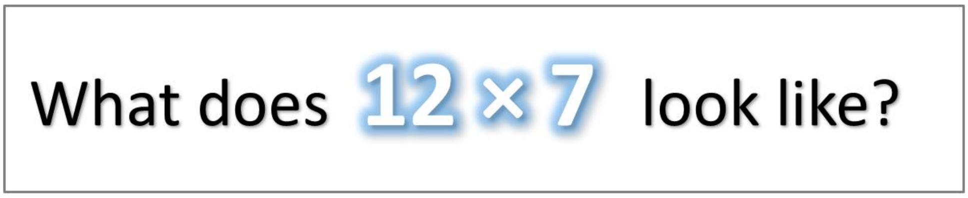


FIGURE 1: The posed question.

The purpose of the activity was for teachers to reflect on the possible ways that one could represent the *structure* of 12 × 7. The teachers responded enthusiastically and produced a wide variety of representations.

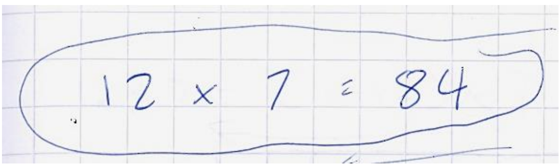


FIGURE 2: Memorisation of multiplication tables.

FIGURE 3: Column multiplication

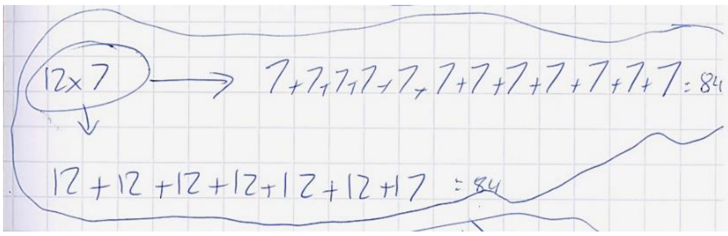


FIGURE 4: Repeated addition.

FIGURE 5: Grouping models (12 groups of 7 or 7 groups of 12).

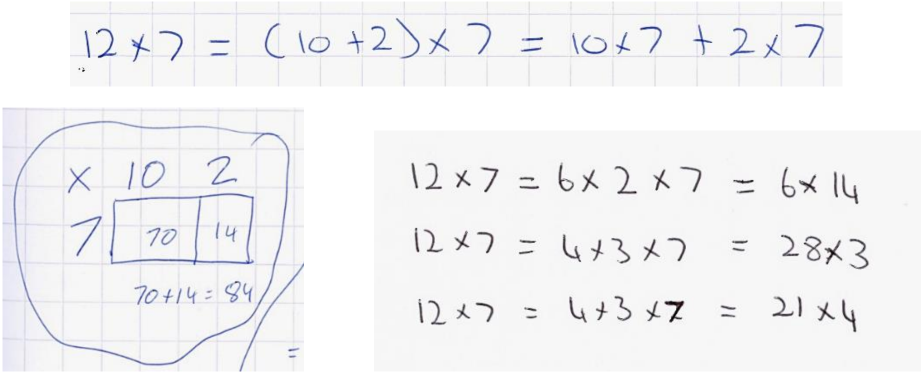
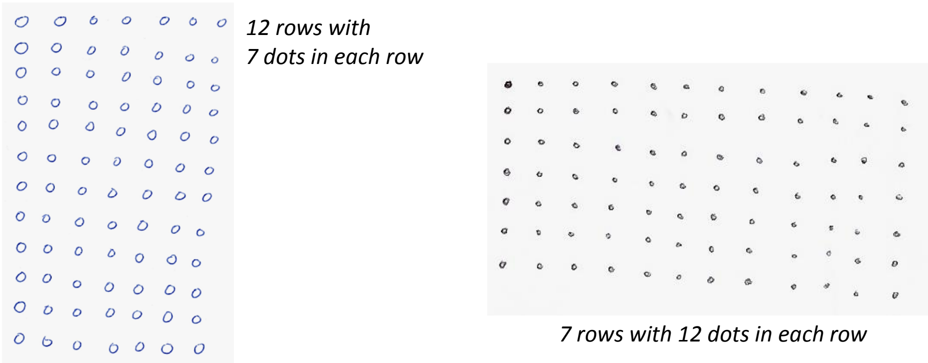


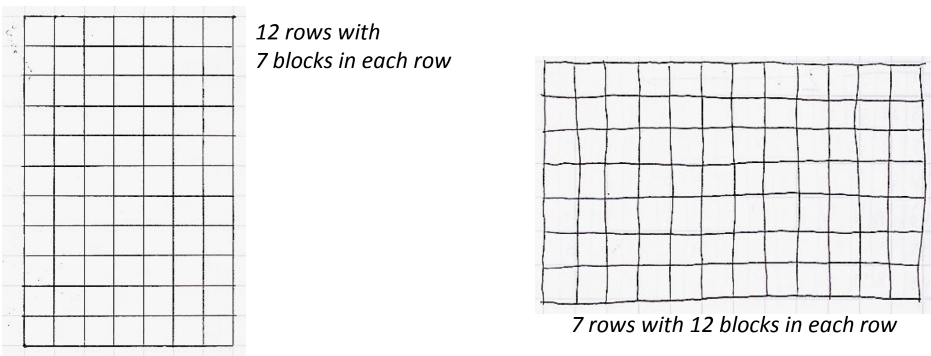
FIGURE 6: Partitioning models.



12 rows with 7 dots in each row

7 rows with 12 dots in each row

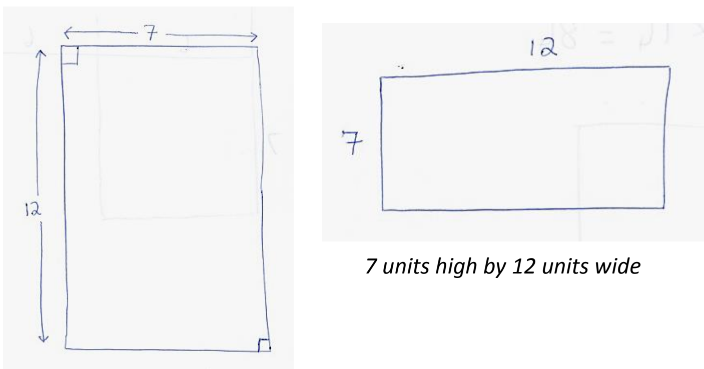
FIGURE 7: Array models.



12 rows with 7 blocks in each row

7 rows with 12 blocks in each row

FIGURE 8: Area models with unit blocks.



7 units high by 12 units wide

12 units high by 7 units wide

FIGURE 9: Area models with measured dimensions.

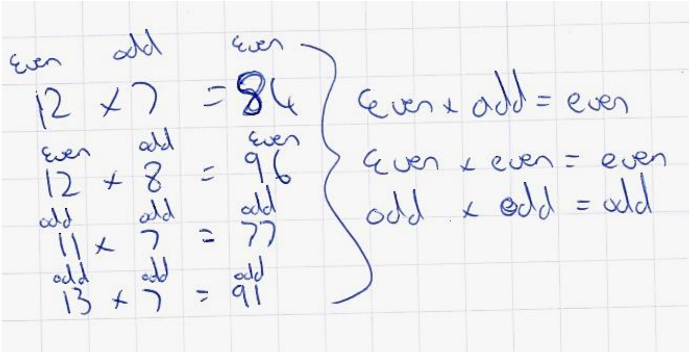


FIGURE 10: Properties of number combinations.

Other representations, not directly explored by the teachers, are also possible:

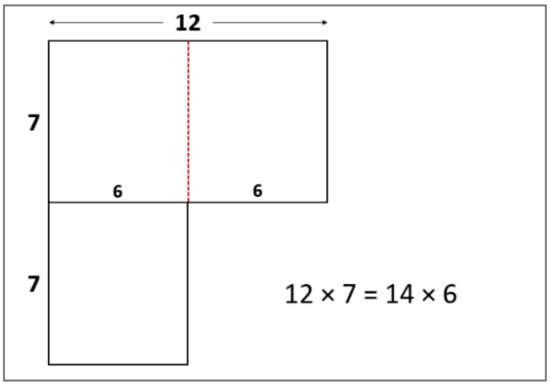


FIGURE 11: Visual representation of the associative property.

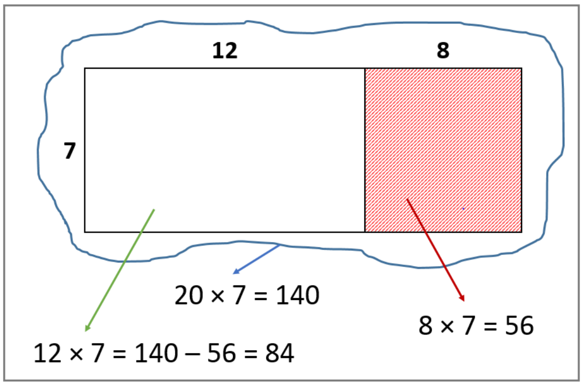


FIGURE 12: Partitioning "up".

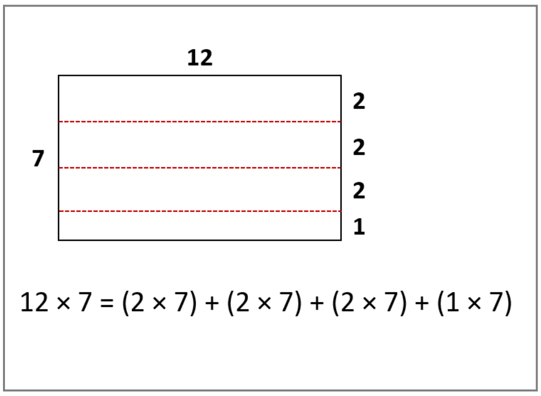


FIGURE 13: Partitioning with multiple partitions.

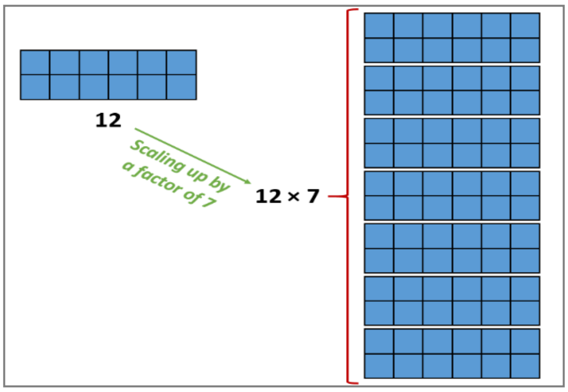


FIGURE 14: Scaling (by a factor of 7).

This multitude of representations opened up the opportunity for in-depth discussion about the use of representations for illustrating mathematical relationships. The key learning points from this discussion are summarised below after an initial discussion about the notion of 'representation'.

WHAT COUNTS AS A REPRESENTATION?

For purposes of this article, a representation encompasses any structure that displays, organises or presents mathematical information. This could be in the form of a mathematical statement, an equation or expression, a formal calculation structure, or a diagram. One way of distinguishing between different types of representations is to use Bruner's (1966) categories of enactive, iconic and symbolic modes. Enactive modes involve physical actions, for example using manipulatives to model mathematical structures (e.g. counters or base ten blocks). Iconic modes are images or graphics that stand for a concept (e.g. graphs, number lines and bar models). Symbolic modes represent mathematical structures via formal mathematical symbols and notation (e.g. number sentences and algebraic expressions). Importantly, these different modes do not operate in a hierarchy with students expected to move from enactive through iconic to symbolic. Rather, the choice of representation is determined by which mode will prove most effective for giving access to a particular mathematical structure. Equally important is giving students access to *all* modes; if we think of each mode as a different puzzle piece, then it is only when all of the puzzle pieces are brought together that the complete picture emerges. For the 12×7 problem in Figure 1, the teachers only provided iconic and symbolic representations (Figures 2 to 14) mainly because I had forgotten to bring manipulatives along to the workshop … eish!

KEY LEARNING POINT NO. 1: WHICH REPRESENTATION AND WHY?

Each representation for 12×7 serves a different purpose, generates a different understanding and imagery of this relationship, and supports a particular way of working. For example, a repeated addition approach (Figure 4) is better supported by picture representations showing repeated groups (Figure 5), and the links between the calculation structure and representation are easy to make. Partitioning methods (Figure 6) are better supported by picture models that illustrate a 'breaking-up' process (Figures 11, 12, 13). Area models with squares (Figure 8) provide an important bridge between array models (Figure 7) and area models involving measured lengths (Figure 9) to support the move away from representations involving a discrete count to an interpretation of geometric space. *Do you and your students understand and agree why you are using a specific representation?*

Food for thought
How deliberate are you when you choose and introduce a representation? Do you have a specific goal in mind for using particular representations? How explicit do you make your choices for your students? Do you choose to introduce representations in a particular order to support a specific learning sequence (e.g. from grouping to partitioning to arrays to area models)? Do you make explicit links between the structures of picture representations and numeric methods (e.g. between partitioning numbers and partitioning models)?

KEY LEARNING POINT NO. 2: DO ALL STUDENTS SHARE THE SAME MENTAL IMAGE?

Using multiple representations provides a useful and powerful strategy for illustrating mathematical relationships and structures, and for providing access to abstract mathematical concepts. However, if we consider that the simple calculation 12×7 yields a multitude of representations and ways of working, then it becomes important to think about whether everyone is operating with the same understanding and mental imagery. *Do all of the students in your class visualise relationships with the same representation as you and as each other?*

Food for thought
Which imagery and perspective dominates how you see this relationship? Do you see it as an area calculation, as an array or count of dots, or perhaps as the number of items in a grouping arrangement? Do you flip casually between different representations? Or, do you make it explicit why you choose to operate with a particular representation and why you may choose to introduce a different representation or more than one representation? Crucially, do your students share the same perspective as you or are they operating from a different visual base?

KEY LEARNING POINT NO. 3: 84 WHAT?

We can all agree that 12 × 7 = 84. But what exactly does this output of 84 represent, and are all 84s the same? For example, are the 84 dots (Figure 7) the same as 84 squares (Figure 8)? Here both 84s represent a specific count of objects (dots and squares), but the 84 squares cover a larger geometric space that the 84 dots and, so, appear to be equal in count only. Then, are the 84 squares equal to the area value of 84 units^2 that describes the 12 × 7 relationship in Figure 9? Both occupy the same 2-dimensional space; however, the relationship 12 × 7 for the squares represents the *discrete count* of squares (arranged in a block with 12 along and 7 down, or 7 along and 12 down), while the relationship 12 × 7 for the rectangle represents the space enclosed between *two measurements* (measured on a *continuous measurement scale*). One representation shows an arrangement of equal sized squares in a particular order, the other shows a fixed shape defined by its measured dimensions. It would be possible to change the arrangement of the squares (for example, into an L-shape or a U-shape) and still have the same count of 84 squares and same area. The same is not possible for the rectangle where the relationship 12 units by 7 units will always define a fixed rectangular shape. *Do your students share the meaning you want them to associate with a specific relationship, and why you are choosing to represent the relationship in a specific way?*

Food for thought
The type of representation used directly affects the meaning or significance attached to a relationship and to the output of that relationship, and a change in a representation will affect the meaning and interpretation prioritised. Do all students in your classroom share the same understanding and meaning of the relationship? Do you know what meaning you want students to understand and do you make it explicit when and why you choose to change that meaning?

KEY LEARNING POINT NO. 4: IS 12 × 7 REALLY EQUAL TO 7 × 12?

Both calculations give an output of 84 – no argument there. However, the structures of 12 × 7 and 7 × 12 look distinctly different depending on the representation used. An arrangement of 12 rows of chairs with 7 chairs in each row will look distinctly different and will fit into a different sized space to an arrangement of 7 rows with 12 chairs in each row. Similarly, 12 plates with 7 biscuits will not give the same experience as 7 plates with 12 biscuits, despite the total number of biscuits (and calories) being the same. The point is that sometimes the flexibility we apply to numerical calculations does not transfer as easily to picture representations of those calculations. Numerically it does not matter if we use the seven or twelve times tables to work out the factors of 84; visually, 7 × 12 and 12 × 7 represent different structures. Partitioning is another case in point here. When applying a partitioning method to work out the answer to the calculation 7 × 12 it does not matter how the 12 is partitioned (e.g. 7 × (10 + 2) or 7 × (2 + 10) or 7 × (6 + 6) or 7 × (7 + 5)) since all variations lead the same answer. Visually, however, the picture representations for each variation show completely different area combinations and represent different arrangements. *When using representations, how explicitly do you link calculation structures to why picture representations of those structures look a certain way?*

Food for thought
Do you link the structures of numerical and algebraic representations to the structures of the picture representations that illustrate those calculations – and discuss similarities and differences? Do you help your students to understand how subtle changes to a relationship directly influences the visual image of that relationship? Do you help your students to recognise the difference between calculating an answer and using a visual image to illustrate the relationship that defines the calculation?

FINAL FOOD FOR THOUGHT

Varied representations provide an essential tool for helping students of all ages to access, visualise and understand mathematical relationships and structures across all topics of the mathematics curriculum. From tables to bar models, number lines, algebraic and statistical graphs and charts, procedural methods, equations and expressions, geometric and trigonometric diagrams, and a host of other schematic diagrams, representations help students organise information and investigate mathematical relationships. However, the use of representations also carries a health warning: every representation carries the potential for enhancing understanding, but also for generating new misconceptions and misunderstandings. Teachers need to be deliberate when using representations. They need to think carefully about why they are relying on particular representations and how and whether a different representation might enhance or confuse understanding. They need to be explicit with students about their choices and motivations for these choices. They need to recognise alternative representations that students might prioritise and the potential for confusion if consistent mental maps and imagery are not agreed. They need to help students understand the connections between relationship structures and specific features of the representations of those structures. Finally, teachers need to dedicate time to helping students develop the skills to construct representations efficiently and accurately, and to discern which representations are most useful in a given situation.

SOME USEFUL READINGS ON REPRESENTATIONS

Bruner, J. S. (1966). *Toward a theory of instruction*. MA: Harvard University Press.

Hoong, L. W., Kin, H. W., & Pien, C. L. (2015). Concrete-Pictorial-Abstract: Surveying its origins and charting its future. *The Mathematics Educator, 16*(1), 1-19.

Mhlolo, M.K., Venkat, H., & Schäfer, M. (2012). The nature and quality of the mathematical connections teachers make. *Pythagoras, 33*(1), Art. #22, 9 pages. http://dx.doi.org/10.4102/pythagoras.v33i1.22.

North, M. (2017). Bars, bars and more bars. *Learning and Teaching Mathematics*, 22, 3-7.

Swan, M. (2008). Designing a multiple representation learning experience in Secondary Algebra. *Educational Designer, 1*(1), 1-17.

Woodward, J., Beckmann, S., Driscoll, M., Franke, M., Herzig, P., Jitendra, A., Koedinger, K. R., & Ogbuehi, P. (2012). *Improving mathematical problem solving in grades 4 through 8: A practice guide (NCEE 2012-4055).* Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education. Retrieved from http://ies.ed.gov/ncee/wwc/publications_reviews.aspx#pubsearch/.