An Elementary Proof of Ptolemy's Theorem

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INTRODUCTION

Ptolemy's theorem is a well-known result in plane geometry and can be stated as follows:

If ABCD is a cyclic quadrilateral with sides $a,\ b,\ c,\ d$ *and diagonals* $e,\ f$, *then* $ac + bd = ef$

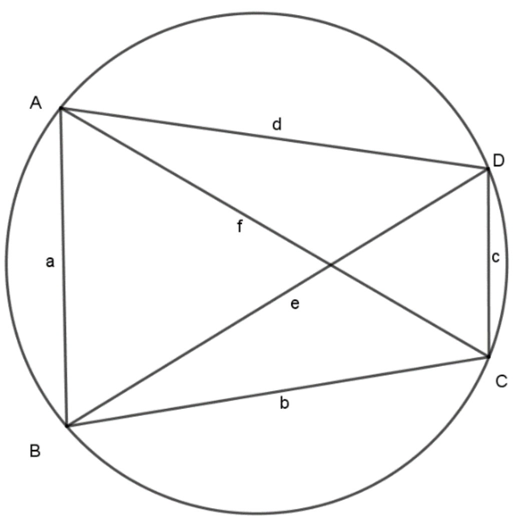

FIGURE 1: Ptolemy's theorem

In this article we present an elementary proof of Ptolemy's theorem based on basic trigonometry, circle geometry and transformation geometry. The proof is different to those described elsewhere – see for example Johnson (1929), Coxeter & Greitzer (1967), Ostermann & Wanner (2012) and Miculita (2017).

A SHORT DIVERSION

Before we get to the proof we first need to establish a useful result that some readers may be unfamiliar with. For a quadrilateral ABCD with diagonals AC and BD, the area of ABCD is half the product of the diagonals multiplied by the sine of the angle between them. With reference to Figure 2:

$$Area\ \Delta ABD = \frac{1}{2} \times BD \times h_1 = \frac{1}{2} \times BD \times AE \sin\theta$$

$$Area\ \Delta BCD = \frac{1}{2} \times BD \times h_2 = \frac{1}{2} \times BD \times CE \sin\theta$$

$$\begin{aligned} Area\ ABCD &= \frac{1}{2} \times BD \times AE \sin\theta + \frac{1}{2} \times BD \times CE \sin\theta \\ &= \frac{1}{2} \times BD \sin\theta\,(AE + EC) \\ &= \frac{1}{2} \times BD \times AC \times \sin\theta \end{aligned}$$

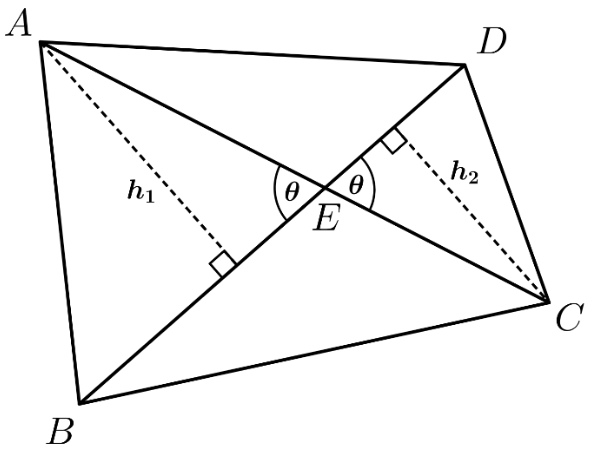

FIGURE 2: $Area\ ABCD = \frac{1}{2} \times BD \times AC \times \sin\theta$

THE PROOF

From the diagram shown in Figure 1, rotate ΔACD by 180° about the center of AC and then reflect it about AC to form a cyclic quadrilateral ABCD' as shown in Figure 3.

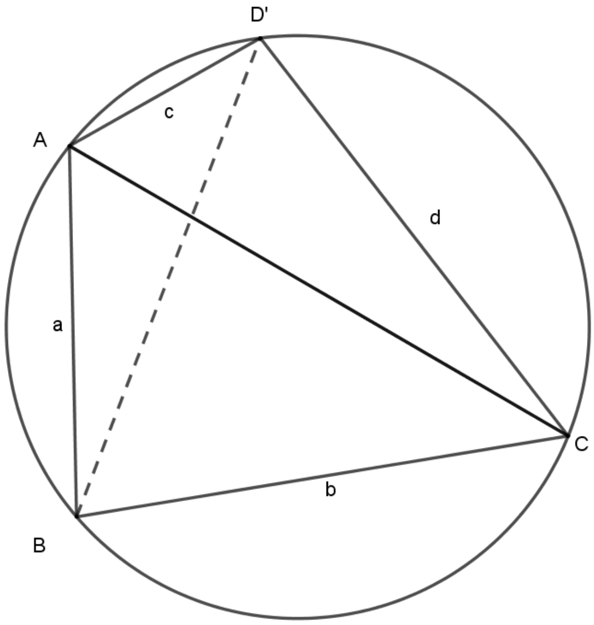

FIGURE 3: Transforming cyclic quadrilateral ABCD

Next label the angles as shown in Figure 4 and Figure 5. From these two diagrams it is easy to deduce that:

$\theta = \alpha + \beta$ (exterior angle of triangle)

$\beta = \beta_1$ (angles in the same segment)

$\phi = \phi_1$ (exterior angle of a cyclic quadrilateral)

$\phi_1 = \alpha + \beta_1 = \theta$

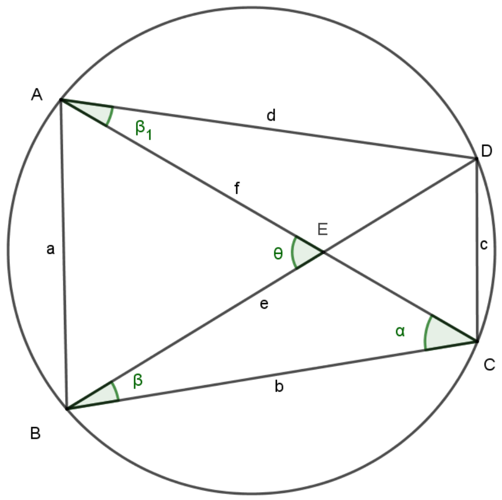

FIGURE 4: Angles in cyclic quadrilateral ABCD

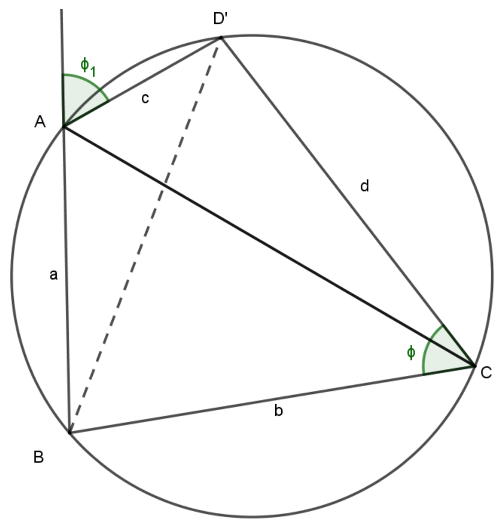

FIGURE 5: Angles in cyclic quadrilateral ABCD'

Finally, since the area of ABCD is equal to the area of ABCD', we have:

$$\frac{1}{2}ef\sin\theta = \frac{1}{2}ac\sin\phi_1 + \frac{1}{2}bd\sin\phi = \frac{1}{2}(ac+bd)\sin\phi = \frac{1}{2}(ac+bd)\sin\theta$$

$$\therefore \ ef = ac + bd$$

REFERENCES

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