Polygons with Numerically Equal Area and Perimeter

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INTRODUCTION

An interesting classroom investigation involves finding polygons with numerically equal area and perimeter. Perhaps the simplest example of such a scenario is a square with side length 4 units which has a perimeter of 16 units and an area of 16 units^2. Another classic example is a 5-12-13 right-angled triangle which has a perimeter of 30 units and an area of 30 units^2. In this article I take these two examples as starting points and expand the discussion into what I hope could be emulated in the classroom as an investigation or guided exploration.

EXPLORING SQUARES

Other than a square of side length 4 units, are there any other squares that have the property that their area and perimeter are numerically equal? Perhaps a good starting point is simply to tabulate area and perimeter for a variety of different squares. Restricting ourselves to integer side lengths for the moment, we could set up a table like this:

Side length (u)	Perimeter (u)	Area (u^2)
1	4	1
2	8	4
3	12	9
4	16	16
5	20	25
6	24	36
7	28	49
8	32	64
9	36	81

TABLE 1: Table of areas and perimeters for squares of different side length

A nice way to do this, particularly if one wanted to include fractional side lengths, is to set up a spreadsheet and enter cell formulas for perimeter and area and then simply copy the formulas down for different side lengths. Even from this short list pupils should hopefully get a sense that once the side length is greater than 4 units the area increases far more rapidly than the perimeter, and the two will never be the same value again. For a side length less than 4 units, pupils should also have a sense that the area is always going to be smaller than the perimeter. A side length of 4 units thus seems to be the only scenario for a square having numerically equal area and perimeter.

Let's see if we can formalise what we have an intuitive sense of from Table 1. For a square with side length x, the perimeter and area will be $4x$ and x^2 respectively. If the area and perimeter are numerically equal then $x^2 = 4x$. This is a quadratic equation which we can rearrange and factorise as follows:

$$x(x-4) = 0$$

The two solutions to this quadratic equation are $x = 0$ and $x = 4$, and since x represents the side length of the square we can ignore the solution $x = 0$, thus confirming our hunch that a side length of 4 units is the only possible solution to the problem.

We could explore this visually by plotting graphs of $y = 4x$ and $y = x^2$ for $x \geq 0$. The visual also confirms our suspicions.

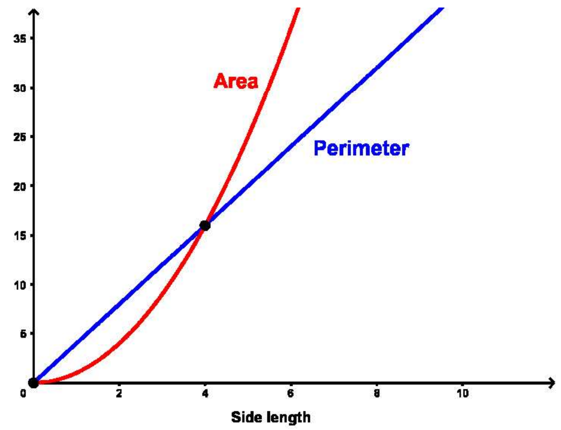

FIGURE 1: Exploring the relation between area and perimeter graphically

EXPLORING RECTANGLES

Now that we have identified a square of side length 4 units as having numerically equal area and perimeter, one might perhaps wonder about rectangles more generally. Are there any other rectangles with integer side lengths that have this property? Let's imagine a rectangle with side lengths x and y. If the area and perimeter are numerically equal then the equation we need to solve is:

$$xy = 2x + 2y$$

Rearranging and making y the subject we have:

$$xy - 2y = 2x$$
$$y(x-2) = 2x$$
$$\therefore \quad y = \frac{2x}{x-2}$$

For integer side lengths we simply need to find integer solutions for both x and y. To start with, it is worth noting that since both x and y are positive, from the above we have the restriction $x > 2$. By substituting in integer values for x we generate the following values for y:

x	3	4	5	6	7	8	9	10
$y = \frac{2x}{x-2}$	6	4	3,333…	3	2,8	2,666…	2,571…	2,5

TABLE 2: Table of related y-values for integer values of x.

In addition to a square with side length 4, the table also reveals a further example – a rectangle with side lengths of 3 units and 6 units, which would have a perimeter of 18 units and an area of 18 units^2. But are there any others?

Notice that since $x > 2$, it will always be true that $2x > x - 2$. This means that as x increases, y will decrease. From Table 2 it looks as though the rate at which y is decreasing is slowing down. One can confirm this, and even get a sense that y will never drop below a value of 2, by setting up a spreadsheet and entering x-values up to 100 for example. If we explore this more algebraically, then we can manipulate the expression for y as follows:

$$y = \frac{2x}{x-2} = \frac{2x-4+4}{x-2} = \frac{2(x-2)+4}{x-2} = 2 + \frac{4}{x-2}$$

Notice that as x increases, the fraction $\frac{4}{x-2}$ gets smaller and smaller and the value of y gets closer and closer to 2. Expressed more formally, as $x \to \infty$, $y \to 2$. This confirms our hunch based on the table.

Using a graphical approach, note that $y = \frac{4}{x-2} + 2$ is simply a hyperbola (Figure 2). The three solutions identified in Table 2 are indicated on the graph. Since there are no integer value of x for $x \in (2; 3)$ and since there are no integer values of y for $y \in (2; 3)$, this confirms that the only two rectangles with integer side lengths that have numerically equal area and perimeter are the 3-by-6 rectangle and the 4-by-4 square. Note however that if we don't restrict ourselves to integer side lengths then every point on the curve represents a solution.

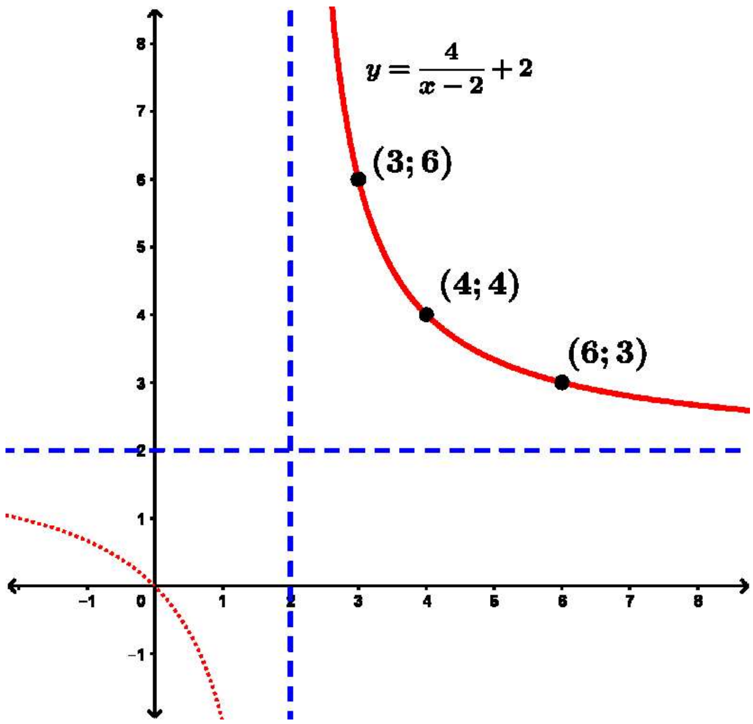

FIGURE 2: Graph of $y = \frac{4}{x-2} + 2$ for $x > 2$

EXPLORING TRIANGLES

As mentioned in the introduction, another classic example of a polygon with numerically equal area and perimeter is a 5-12-13 right-angled triangle, which has a perimeter of 30 units and an area of 30 units^2. Are there any other right-angled triangles, with integer side lengths, that have this property? Let's imagine a right-angled triangle with perpendicular side lengths a and b. The hypotenuse of this triangle would be $\sqrt{a^2+b^2}$, and its area would be $\frac{1}{2}ab$. If the area and perimeter were numerically equal, then we would need to find integer solutions to the equation:

$$a + b + \sqrt{a^2+b^2} = \frac{1}{2}ab$$

While doable, this is going to be a somewhat cumbersome process. Interested readers are nonetheless encouraged to have a go. A slightly more elegant approach to the problem is to make use of the fact that Pythagorean triples a, b, c can be generated from:

$$a = m^2 - n^2 \quad ; \quad b = 2mn \quad ; \quad c = m^2 + n^2$$

where m and n are positive integers with $m > n$. We thus have:

$$\text{Perimeter} = a + b + c = 2m^2 + 2mn$$

$$\text{Area} = \frac{1}{2}ab = mn(m^2 - n^2)$$

If the area and perimeter were numerically equal, then we would need to find integer solutions to the equation:

$$2m^2 + 2mn = mn(m^2 - n^2)$$

$$\therefore \quad 2m(m+n) = mn(m+n)(m-n)$$

Since m and n are both positive integers, we can divide through by m and $(m+n)$ since neither can equal zero. Thus:

$$2 = n(m-n)$$

This can be rearranged to give:

$$m = n + \frac{2}{n}$$

Now, since m and n are both positive integers, n can only take on values 1 and 2, with an associated value of $m = 3$ in both cases. When $n = 1$ and $m = 3$ the Pythagorean triple a, b, c generated is 8,6, 10. When $n = 2$ and $m = 3$ the Pythagorean triple a, b, c generated is 5,12,13. These are thus the only right-angled triangles with integer side lengths for which the area and perimeter are numerically equal.

EXPLORING REGULAR POLYGONS

We have identified a 4-by-4 square as having numerically equal area and perimeter. What about other regular polygons? Let us loosen the restriction of integer side lengths and look more generally at the required conditions for a regular polygon to have this property. To begin with, let's consider a regular pentagon with side length x.

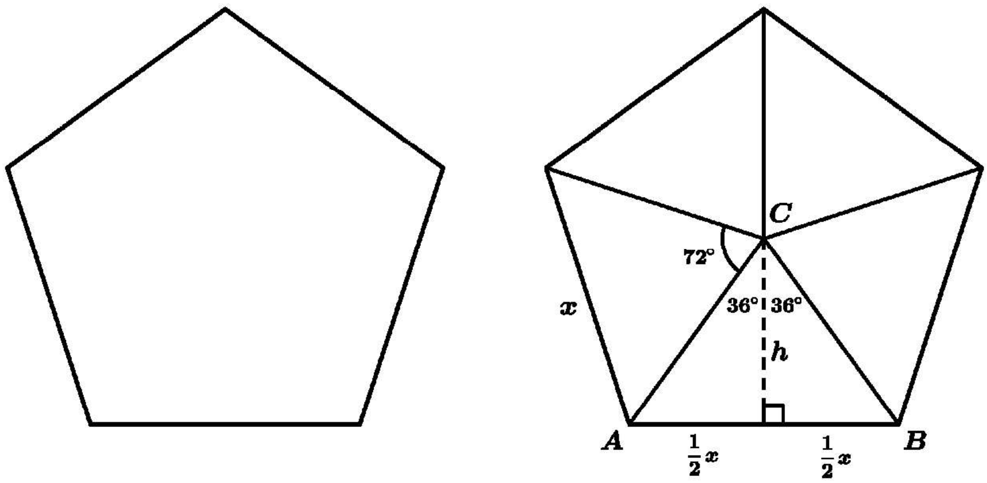

FIGURE 3: Regular pentagon with side length x

If we subdivide the pentagon into five identical isosceles triangles, then the area of each of these triangles is:

$$Area\ \Delta ABC = \frac{1}{2}(x)(h)$$

Since $\tan 36° = \left(\frac{1}{2}x\right)/h$ we have $h = \left(\frac{1}{2}x\right)/(\tan 36°)$. We can thus write:

$$Area\ \Delta ABC = \frac{1}{2}x \times \frac{\frac{1}{2}x}{\tan 36°} = \frac{x^2}{4\tan 36°}$$

$$\therefore\ Area\ of\ pentagon = \frac{5x^2}{4\tan 36°}$$

If the area and perimeter are numerically equal, then:

$$\frac{5x^2}{4\tan 36°} = 5x$$

We thus have $x = 4\tan 36°$.

If we follow a similar process for a hexagon with side length x, then the perimeter of the hexagon is $6x$ and the area of the hexagon is $\frac{6x^2}{4\tan 30°}$. Equating the two expressions and solving for x gives $x = 4\tan 30°$. If we generalise this to an n-sided regular polygon then the side length for which the perimeter and area are numerically equal is:

$$x = 4\tan\left(\frac{180°}{n}\right)\ ;\ n \geq 3, n \in \mathbb{Z}$$

If we test this formula for a square (i.e. $n = 4$) then $x = 4\tan\left(\frac{180°}{4}\right) = 4\tan 45° = 4$ as previously obtained. Plotting the graph of $y = 4\tan\left(\frac{180°}{n}\right)$ shows how the side length decreases as the number of sides increases.

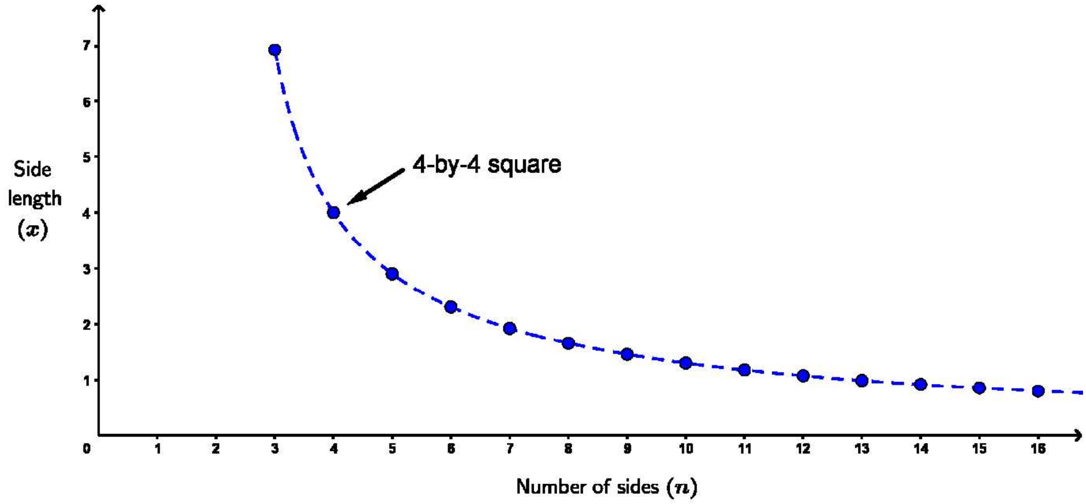

FIGURE 4: Graph of number of sides versus side length

Note that as $n \to \infty$, $\frac{180°}{n} \to 0$ and $\tan\left(\frac{180°}{n}\right) \to 0$. The limiting case is the circle which would have numerically equal area and perimeter when $2\pi r = \pi r^2$, i.e. when the circle has a radius of 2 units. This value of $r = 2$ is not without significance. We have established that for a regular n-sided polygon to have numerically equal area and perimeter the side length must be $x = 4\tan\left(\frac{180°}{n}\right)$. Furthermore, from Figure 3 we see that in general we have:

$$\tan\left(\frac{180°}{n}\right) = \frac{\frac{1}{2}x}{h} \quad \Rightarrow \quad h = \frac{\frac{1}{2}x}{\tan\left(\frac{180°}{n}\right)}$$

$$\therefore \quad h = \frac{\frac{1}{2} \times 4\tan\left(\frac{180°}{n}\right)}{\tan\left(\frac{180°}{n}\right)} = 2$$

The perpendicular height of each isosceles triangle is thus 2, independent of how many sides the polygon has. If one hasn't come across this result before, it is rather surprising. In essence it means that all regular polygons that have the property that their area and perimeter are numerically equal have an inscribed circle with a radius of 2 units.

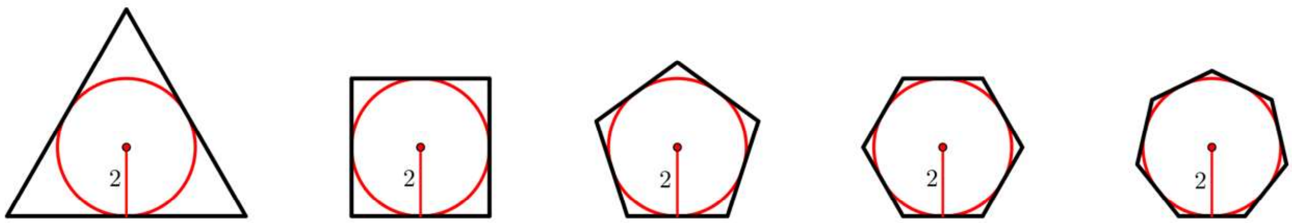

FIGURE 5: Regular polygons with numerically equal area and perimeter

One can arrive at this result in a far more direct way as follows. With reference to Figure 6, note that each triangular subdivision has area $\frac{1}{2}xh$. In general, for an n-sided regular polygon with side length x, the polygon would have area equal to $\frac{1}{2}nxh$ and a perimeter equal to nx. Equating these two expressions yields:

$$\frac{1}{2}nxh = nx$$

And since $n \neq 0$ and $x \neq 0$ we can divide both sides by nx:

$$\therefore \quad \frac{1}{2}h = 1$$

$$\therefore \quad h = 2$$

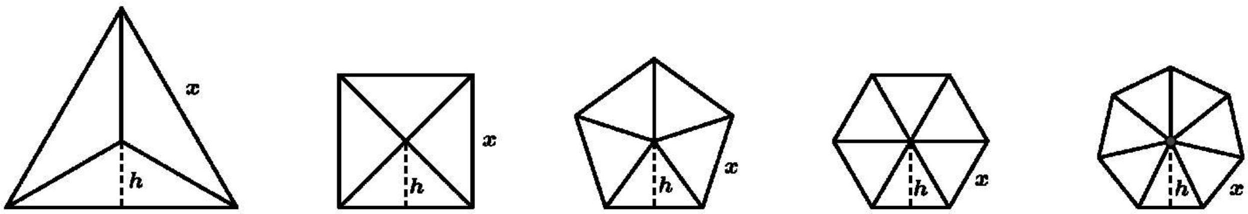

FIGURE 6: Regular n-sided polygons with side length x subdivided into n isosceles triangles

CONCLUDING COMMENTS

Two-dimensional shapes that have numerically equal area and perimeter are popularly known as *equable* or *perfect* shapes. A great deal has been written about equable shapes over the years, and they remain a source of fascination. What I hope I have illustrated in this article is how a simple starting point – the observation that a square with side length of 4 units has numerically equal area and perimeter – can be developed into a classroom investigation suitable for a wide range of age groups and mathematical ability. From an initial process of simply tabulating side length, perimeter and area for different squares, the scope was extended to include other rectangles and right-angled triangles with integer side lengths, and finally regular polygons. In doing so we were able to touch on aspects of number theory, algebraic reasoning, and graphical representations of functions.

There is a great deal more that one could explore in relation to equable shapes, but as a topic for classroom investigation I think what I have included in this article has the potential to encourage rich and meaningful discussion as well as further independent exploration. The amount of scaffolding and guidance pupils will require will vary, and the depth to which one might go would of course depend on the grade of the class. However, since the task is so wonderfully divergent, able pupils can always forge ahead on their own, and this aspect of the investigation is particularly useful for mixed ability classes.