Rotating a Square and Rectangle

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Imagine a unit square positioned in the first quadrant of the Cartesian plane as illustrated in Figure 1. The four vertices of the square are the points $(0;0)$, $(0;1)$, $(1;1)$ and $(1;0)$.

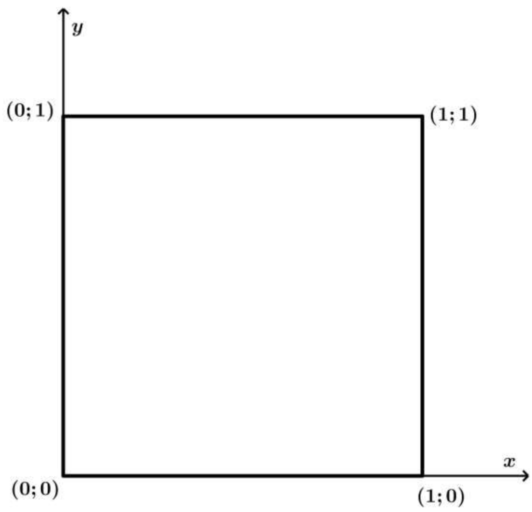

FIGURE 1: A unit square positioned in the first quadrant

Now imagine rotating the square in a clockwise direction such that one of the vertices is always in contact with the x-axis, and another is always in contact with the y-axis. What curve would be traced out by the top left vertex, originally at $(0;1)$, as the square rotates through a quarter turn? Figure 2 illustrates this process and traces the locus of the vertex in question.

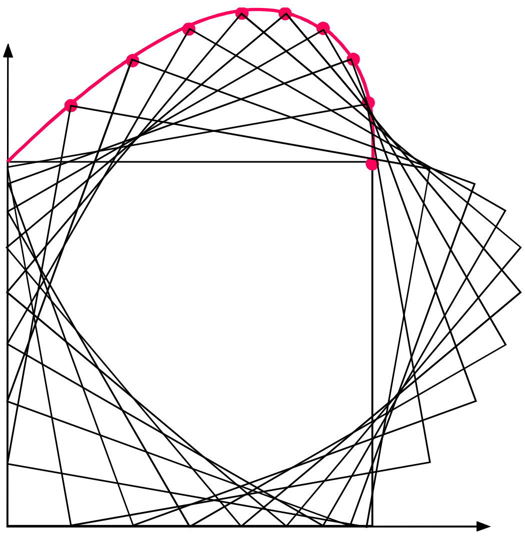

FIGURE 2: Rotating a square through a quarter turn

Would it be possible to determine an explicit formula for the path illustrated in Figure 2? Let's consider a general unit square in the process of rotating, as illustrated in Figure 3. The vertex tracing out the path we are interested in is $A(x; y)$. The vertex in contact with the x-axis is $E(p; 0)$, and the vertex in contact with the y-axis is $C(0; q)$.

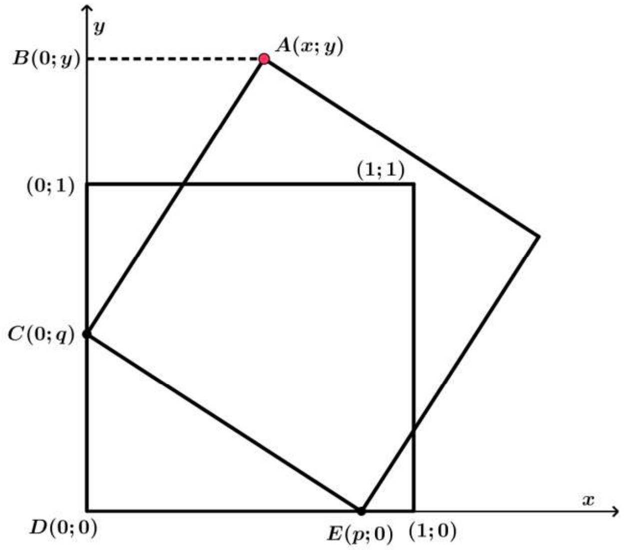

FIGURE 3: Generalising the rotating square

Since the side length of the square is 1 unit, it is clear that $p^2 + q^2 = 1$. From this we can express p in terms of q as $p = \sqrt{1 - q^2}$. Since triangles ABC and CDE are similar (in fact they are congruent), it follows that:

$$\frac{y - q}{p} = \frac{x}{q} = 1$$

From this we can write $y = \left(\frac{p}{q}\right) x + q$ and note that $q = x$. Substituting $p = \sqrt{1 - q^2}$ into $y = \left(\frac{p}{q}\right) x + q$ gives $y = \left(\frac{\sqrt{1 - q^2}}{q}\right) x + q$. Finally, since $q = x$, we can write $y = x + \sqrt{1 - x^2}$. The graph of this function is illustrated in Figure 4.

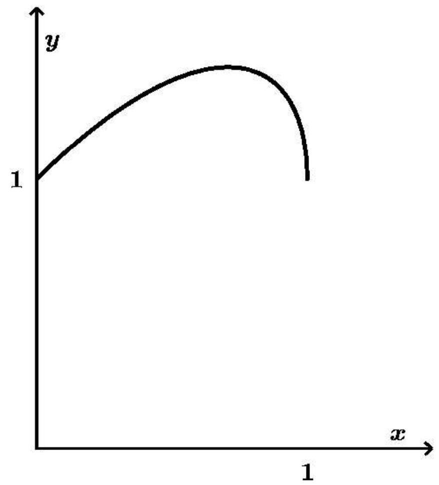

FIGURE 4: The graph of $y = x + \sqrt{1 - x^2}$ for $x \geq 0$

Let us now extend this idea to a more general rectangle. Consider a rectangle with length 1 and width w. As in the case of the square, what path would be traced out by the top left vertex as the rectangle rotates through a quarter turn? Figure 5 illustrates the locus of the vertex in question.

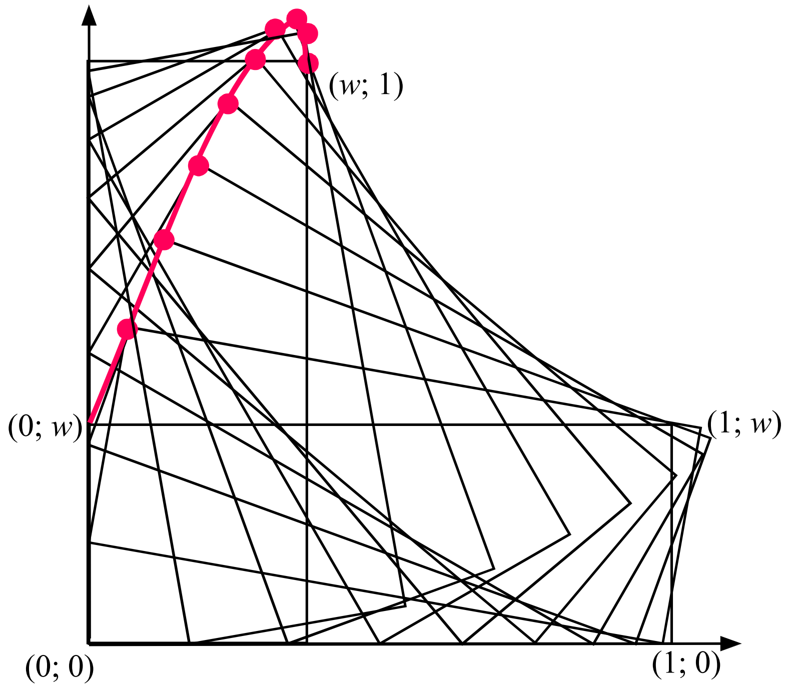

FIGURE 5: Rotating a rectangle through a quarter turn

As with the square, let us try to establish an explicit formula for the path illustrated in Figure 5. With reference to Figure 6, the vertex tracing out the path we are interested in is $A(x; y)$. The vertex in contact with the x-axis is $E(p; 0)$, and the vertex in contact with the y-axis is $C(0; q)$.

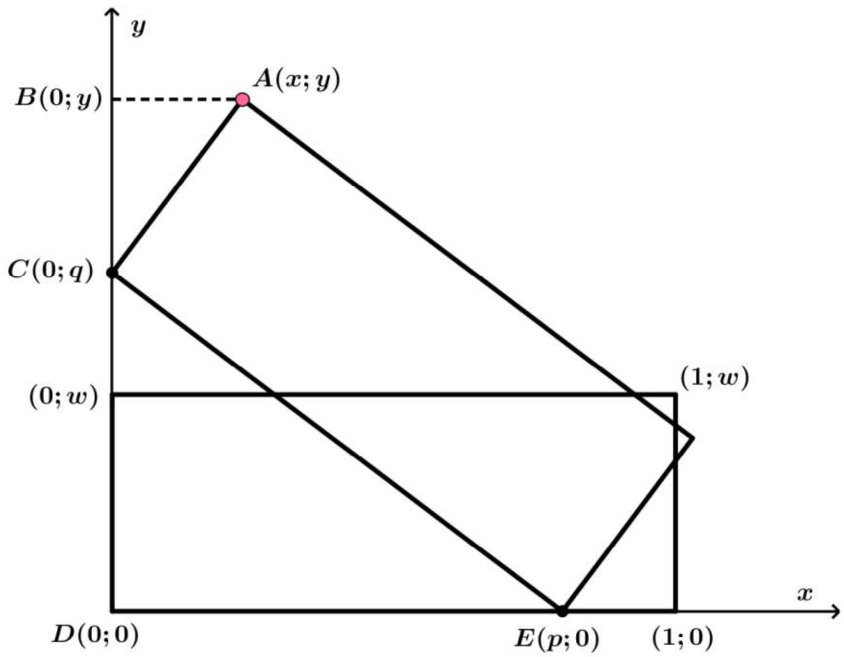

FIGURE 6: Generalising the rotating rectangle

Since the length of the rectangle is 1 unit, it is clear that $p^2 + q^2 = 1$. From this we can express p in terms of q as $p = \sqrt{1 - q^2}$. Since triangles ABC and CDE are similar, it follows that:

$$\frac{y - q}{p} = \frac{x}{q} = \frac{w}{1}$$

From this we can write $y = \left(\frac{p}{q}\right)x + q$ and see that $q = \frac{x}{w}$. Substituting $p = \sqrt{1 - q^2}$ into $y = \left(\frac{p}{q}\right)x + q$ gives $y = \left(\frac{\sqrt{1 - q^2}}{q}\right)x + q$. Finally, since $q = \frac{x}{w}$, we can write:

$$y = \left(\frac{\sqrt{1 - \left(\frac{x}{w}\right)^2}}{\frac{x}{w}}\right)x + \frac{x}{w}$$

This we can simplify to $y = \frac{x}{w} + w\sqrt{1 - \left(\frac{x}{w}\right)^2}$. Figure 7 illustrates graphs of this function for different values of w, for $x \geq 0$.

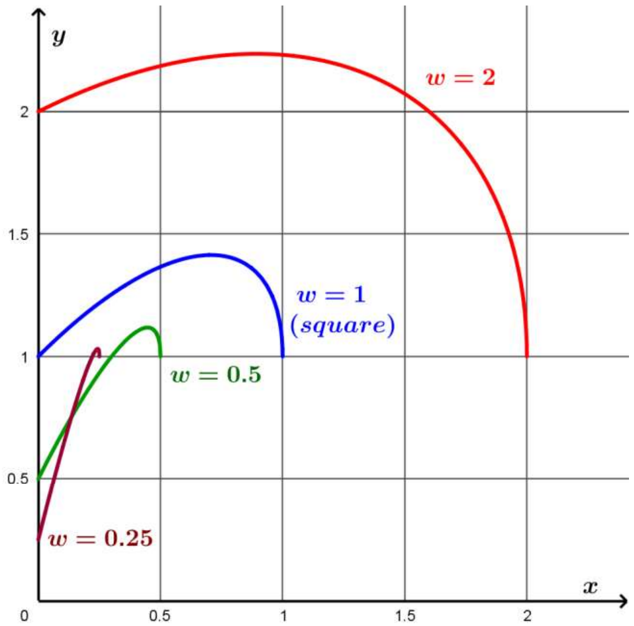

FIGURE 7: Graphs for rectangles with different widths

Note that as w gets smaller the greater the initial gradient of the slope. In each case the function increases to some maximum value and then ends with a "hook". One could readily use calculus to determine the point at which the function is a maximum, but our intuition suggests that the maximum is reached when the diagonal of the rectangle is vertical, meaning the maximum height is $\sqrt{w^2 + 1}$. It is left for the reader to verify this.

ACKNOWLEDGEMENT

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