The Value of using Signed Quantities in Geometry

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INTRODUCTION

In the previous issue of *Learning and Teaching Mathematics,* it was shown in De Villiers (2020) how a formula for the area of a quadrilateral in terms of its diagonals and the sine of the angle between them can easily be extended to concave or crossed quadrilaterals with the use of 'signed' or 'directed' areas.

While the idea of negative quantities is familiar, and taken for granted in arithmetic, algebra, trigonometry, and even in physics, the mere mention of concepts in geometry like 'negative' areas, distances or angles often generates incredulous looks among people who hear of it for the first time. Of course, the ancient Greeks never considered the possibility of negative quantities in geometry, but at least since about the 19th century, geometers have gradually grown accustomed to this 'revolutionary' idea, mainly because of its value in providing one general proof, rather than having to consider and prove several different cases.

The purpose of this article is to give two further examples of the value of directed or signed quantities in geometry, focusing this time on directed angles and directed distances.

FIRST EXAMPLE: DIRECTED ANGLES

The first example is a well-known theorem prescribed in most school mathematics curricula around the world, including the South African High School Mathematics Curriculum, where it is usually dealt with in Grade 11/12.

Theorem: The angle subtended by an arc (or chord) at the centre of a circle is twice the size of the angle subtended by the same arc (or chord) at the circumference of the circle (on the same side of the chord as the centre).

Proof: For a complete proof of the result, one has to consider three different geometric configurations as shown in Figure 1.

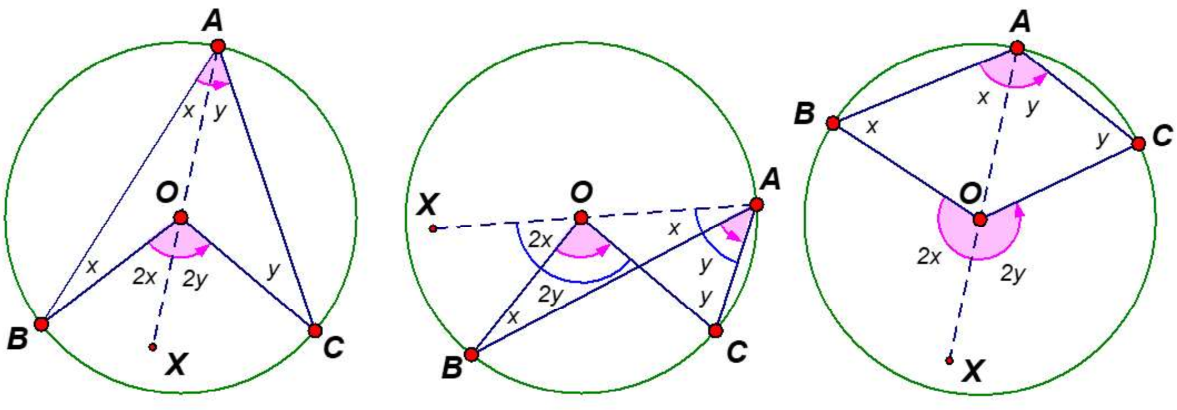

FIGURE 1: Three different cases for the 'Angle at Centre' theorem

When proving the result, teachers and textbooks sometimes only consider the first case in Figure 1, but this is not sufficient. The proof of the third case is identical to the first case, although students often have difficulty in visualizing and correctly applying the theorem when the central angle becomes reflexive. However, the proof of the second case is quite different from the other two.

The proofs for the first and third cases in Figure 1 are the same. For example, draw *AO* and extend to *X* as indicated. Then:

$\angle OBA = x = \angle OAB \ldots OB = OA;$ radii

$\Rightarrow \angle BOX = 2x \ldots$ exterior angle of ΔABO

$\angle OCA = y = \angle OAC \ldots OC = OA;$ radii

$\Rightarrow \angle XOC = 2y \ldots$ exterior angle of ΔAOC

Thus, $\angle BOC = \angle BOX + \angle XOC = 2x + 2y = 2(x+y) = 2\angle BAC$

However, the second case is different, and involves the subtraction of angles. Although the first part of the proof is identical, the difference is apparent in the last step:

$\angle OBA = x = \angle OAB \ldots OB = OA;$ radii

$\Rightarrow \angle BOX = 2x \ldots$ exterior angle of ΔABO

$\angle OCA = y = \angle OAC \ldots OC = OA;$ radii

$\Rightarrow \angle XOC = 2y \ldots$ exterior angle of ΔAOC

Thus, $\angle BOC = \angle XOC - \angle BOX = 2y - 2x = 2(y-x) = 2\angle BAC$

Using the idea of 'directed' (or 'signed') angles, however, one does not need to write down the second proof, and the first proof suffices for all three cases, provided one clearly states at the beginning that one is making use of directed angles.

The idea is really quite simple. Note that in the first and third cases the size of $\angle BOX$ is determined by an *anti-clockwise* rotation of ray *OB* around *O* to map onto ray *OX* (which in trigonometry is normally defined as a positive rotation). However, in the second case the size of $\angle BOX$ is determined by a *clockwise* rotation of ray *OB* around *O* to map onto ray *OX* (which in trigonometry is normally defined as a negative rotation). In other words, in the second case, $\angle BOX$ can be viewed as negative in relation to $\angle BOX$ in the other two cases. Most significantly, the first proof therefore holds provided we regard $\angle BOX$ as negative in the second case, as the first proof then automatically covers the necessary subtraction in the second case.

One of the advantages therefore of using directed angles is that it avoids having to write down several different proofs in order to cover different cases. It is important, however, that one clearly states at the outset of such a proof that one is assuming directed angles.

Most dynamic geometry packages allow for the measurement of directed angles. Although the default setting of angle measurement in *Sketchpad* is for the absolute value of an angle, this can easily be changed in the Edit/Preferences/Units/Angle menu to 'directed degrees'. The popular GeoGebra actually has 'directed degrees' as its default measurement. For example, when a quadrilateral *ABCD* is changed into a crossed quadrilateral (as shown in Figure 2) by dragging vertex *C* across *AD*, two reflex angles are formed at the vertices *C* and *D*, and the angle sum of the angles of the crossed quadrilateral becomes 720°.

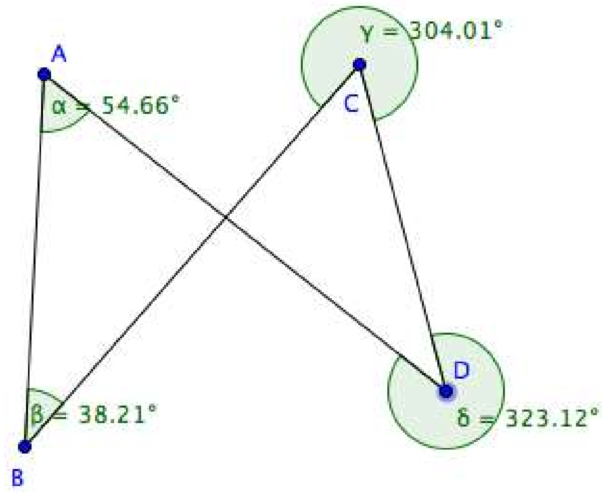

FIGURE 2: The angles of a crossed quadrilateral in GeoGebra

Instead of the usual practice of 'monster-barring' crossed quadrilaterals, such a surprising empirical discovery about crossed quadrilaterals by learners could create an excellent opportunity for not only learning about directed angles, but also about explaining why (proving that) the angle sum is 720° (De Villiers, 2003, pp. 40-44; 156-157).

SECOND EXAMPLE: DIRECTED DISTANCES

Consider the following problem from De Villiers (2003, pp. 26, 149-150), which is easily accessible for learners in Grades 8-9: Prove that the sum of the distances[4] from a point to the sides of a parallelogram is constant. A dynamic geometry sketch is available online for the reader or learners to explore:

http://dynamicmathematicslearning.com/parallelogram-distances.html

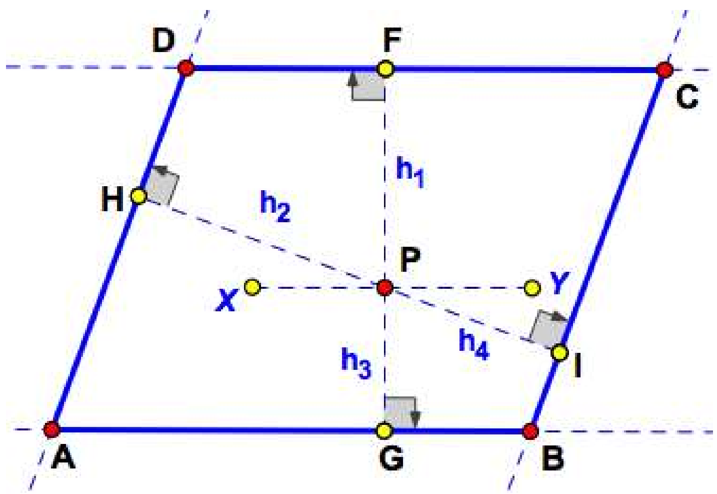

FIGURE 3: Distances to the sides of a parallelogram.

[4] With 'distances' here is meant 'shortest distances', which are the perpendiculars from *P* to the sides.

Proof: Consider Figure 3 showing parallelogram *ABCD* with an arbitrary point *P*, and the distances from *P* to the sides. Although in the accurately drawn diagram in Figure 3 it is clear that *FPG* and *HPI* are straight lines, we may not assume they are straight – we need to prove[5] it. Draw line *XPY* parallel to the opposite sides *AB* and *CD*. Then it follows that:

$$\angle XPF = 90° = \angle DFP \text{ (co-interior } \angle s)$$

$$\angle XPG = 90° = \angle BGP \text{ (alternate } \angle s)$$

$$\Rightarrow \angle XPF + \angle XPG = 90° + 90° = 180°$$

Hence, *FPG* is a straight line. In the same way, it can be shown that *HPI* is also straight. Now note that:

$$h_1 + h_3 = \text{constant} = c_1 \ldots \text{ distance between parallels } AB \text{ and } CD \text{ is constant}$$

$$h_2 + h_4 = \text{constant} = c_2 \ldots \text{ distance between parallels } AD \text{ and } BC \text{ is constant}$$

$$\Rightarrow h_1 + h_2 + h_3 + h_4 = c_1 + c_2 = \text{constant.}$$

This completes the proof.

What happens when point *P* is dragged outside the parallelogram *ABCD*? The reader is requested to explore this now in the dynamic sketch at the URL provided earlier.

While the sum of the distances remains constant as long as *P* is inside *ABCD*, the reader will find as soon as *P* is moved outside *ABCD,* as shown in Figure 4, that this sum is no longer constant. So is the theorem only valid while *P* is inside the parallelogram?

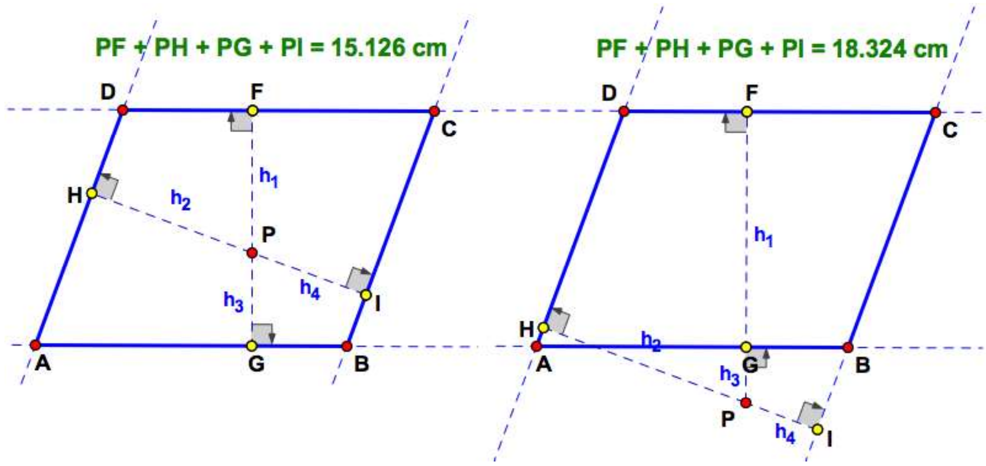

FIGURE 4: Moving point *P* outside the parallelogram

If we use *directed distances* (or *vectors*) it is easy to see that the result actually still holds when *P* is moved outside *ABCD*, because it results in a change of direction for some of the distances. Consider Figure 5, which shows vectors $\overrightarrow{GP}$ and $\overrightarrow{PF}$ with the resultant of $\overrightarrow{GP} + \overrightarrow{PF} = \overrightarrow{GF}$. Note that in the second case, when *P* is moved outside the parallelogram, $\overrightarrow{GP}$ has changed direction and the magnitude of $\overrightarrow{GP}$ now needs to be subtracted from the magnitude of $\overrightarrow{PF}$. Hence, the resultant $\overrightarrow{GF}$ remains unchanged[6].

[5] Pedagogically, when proving it in class, it is probably advisable rather to make use of an inaccurately drawn sketch so that these lines do NOT appear straight.

[6] Sometimes instead of vectors, the convenient definition is used that all distances falling completely outside a figure are regarded as negative. For example, note that *PG* changes direction as soon as *P* is moved outside the parallel lines, and can therefore be considered as negative.

Since the same argument applies for the sum of the directed distances (vectors) $\overrightarrow{IP}$ and $\overrightarrow{PH}$, the total sum of the directed distances (vectors) from P to the sides remains constant, even when P is moved outside $ABCD$.

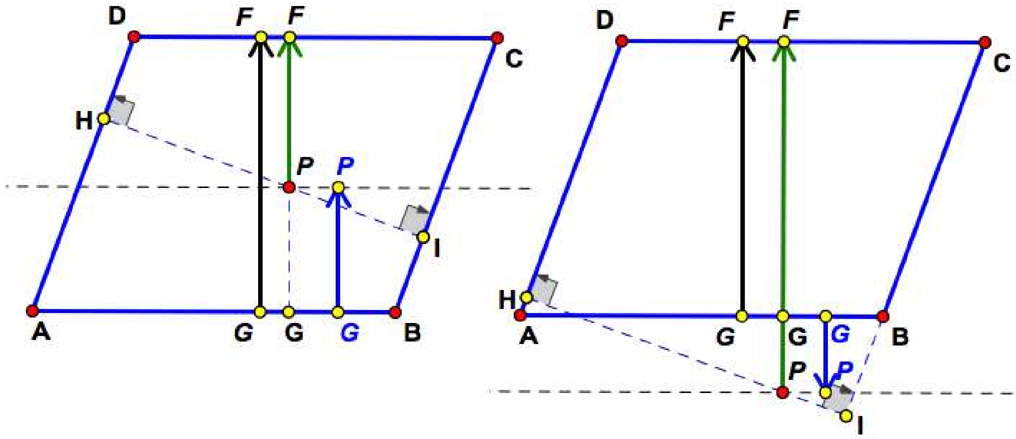

FIGURE 5: Using directed distances/vectors

As we have seen in this example, the value of using directed distances (or vectors) in geometry is that many results can be extended so that they hold more generally. Viviani's theorem, for example, that states that the sum of the distances from a point to the sides of an equilateral triangle is constant (see Samson, 2012), can similarly be generalized by using directed distances so that it is also valid when the point is moved outside the triangle. This also applies to generalizations of Viviani's theorem to equi-angled and equilateral polygons, to Clough's variation of Viviani's theorem (De Villiers, 2012), as well as to generalizations of Viviani to 3D (De Villiers, 2013).

Finally, since this result for a parallelogram follows from the 'constant distance' property of pairs of parallel lines, it is easy to see that it generalizes to any 2n-gon with opposite sides parallel. It therefore provides another good example of the so-called 'discovery' function of proof mentioned in De Villiers (1990), where an explanatory proof leads to further generalization.

CONCLUDING REMARKS

This paper has given two examples of the value of using directed areas and distances. In the first case with directed angles, it was useful because writing down a 'directed angles' proof would automatically cover the three different configurations. In the case with directed distances, it was useful to extend the result to points outside the parallelogram. Learning about directed quantities in geometry might be particularly useful for learners who wish to participate in high-level mathematics competitions (like PAMO, SAMO and IMO). It would also be a suitable topic to address in a Mathematics Club for talented learners at a school.

REFERENCES

De Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras* 24, 17-24.

De Villiers, M. (2003). *Rethinking Proof with Sketchpad*. Emeryville: Key Curriculum Press.

De Villiers, M. (2012). An example of the explanatory and discovery function of proof. *Pythagoras*, 33(3), Art. #193, http://dx.doi.org/10.4102/pythagoras.v33i3.193

De Villiers, M. (2013). 3D generalizations of Viviani's Theorem. *The Mathematical Gazette*, 97(540), pp. 441-445.

De Villiers, M. (2020). The area of concave and crossed quadrilaterals in terms of their diagonals. *Learning and Teaching Mathematics*, *28*, pp. 19-21.

Samson, D. (2012). Viviani's Theorem – a geometrical diversion. *Learning and Teaching Mathematics*, *13*, pp. 28-32.