Making my Unimportant Mathematical Discovery Count

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The discovery

One night I was sitting at my desk preparing the next day's lecture of a *Functions and Algebra* course for pre-service maths teachers. We were dealing with the quadratic function at the time and in the next session I was planning to focus on the different forms of the quadratic equation [i.e. the general form $y = ax^2 + bx + c$; the turning point form: $y = a\,(x - p)^2 + q$; and the "root form": $y = a\,(x - \alpha)\,(x - \beta)$ where α and β are roots] and the different information that each equation gives us. Although this work is covered in Grade 11 mathematics, I knew from experience that students would benefit from revisiting aspects like *completing the square* and the origins of the quadratic formula.

While sitting doing my prep, I began to wonder about the relationship between the forms

$$y = a\,(x - p)^2 + q \quad \text{and} \quad y = a\,(x - \alpha)\,(x - \beta).$$

I had never thought about this before, and that very fact concerned me. *Why* had I not thought about it? I continued to explore, using both algebra and Geometer's Sketchpad (GSP), and this is what I found. I started with a quadratic in root form:

$$f\,(x) = (x - 1)\,(x + 3)$$

When I graphed it, I saw the axis of symmetry was $x = -1$ and the minimum value was $y = -4$ (see fig. 1). Looking at the graph, I saw that if I shifted it up 4 units, it would have equal roots at $x = -1$. The equation of this new graph would be $y = f(x) + 4$, i.e. $y = (x - 1)(x + 3) + 4$.

Manipulating the new equation, I got

$$\begin{aligned} y &= x^2 + 2x - 3 + 4 \\ &= x^2 + 2x + 1 \\ &= (x + 1)^2 \end{aligned}$$

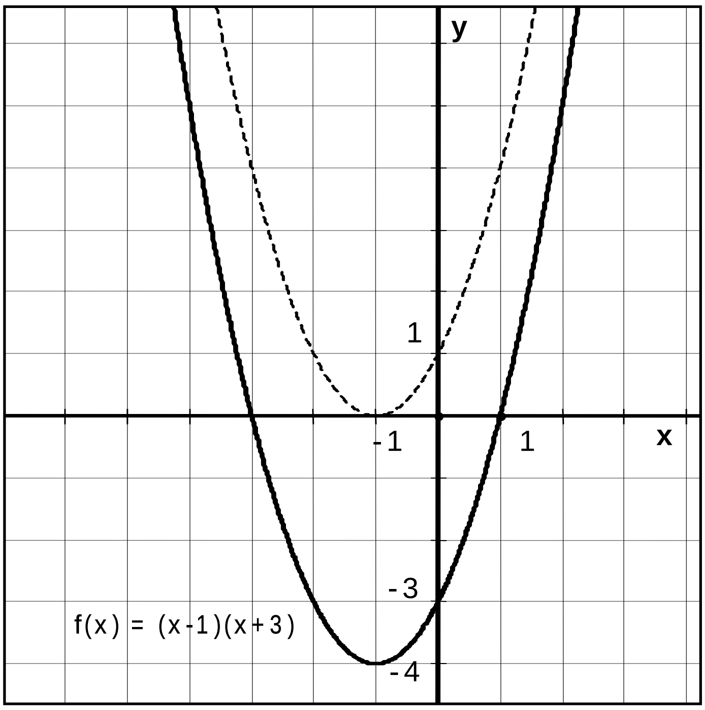

Fig. 1 Graph of $f\,(x) = (x-1)(x+3)$ and its translation

This form (i.e. $y = a\,(x - p)^2 + q$) tells me the axis of symmetry is $x = -1$ and the graph turns on the x-axis since $q = 0$. Clearly this means the graph has equal roots at $x = -1$ since it is in the form $y = (x + 1)(x + 1)$.

I then took an algebraic approach, starting with $y = (x - 1)(x + 3)$ and completed the square.

$y = x^2 + 2x - 3$
$= (x^2 + 2x + 1) - 1 - 3$
$= (x + 1)^2 - 4$

This confirmed what I already knew: that the original graph has a turning point at (–1; –4). But it also shows clearly the translation that I performed to get the new graph: if I translate the original graph 4 units upwards, the equation becomes $y = (x + 1)^2 - 4 + 4$ or $y = (x + 1)^2$.

I summarise this now for a more general quadratic $y = (x - \alpha)(x - \beta)$.

$y = (x - \alpha)(x - \beta)$ (I have chosen $a = 1$, to keep the manipulation simple)

$= x^2 - (\alpha + \beta)x + \alpha\beta$ (expanding factors)

$= (x - \frac{\alpha+\beta}{2})^2 + \frac{4\alpha\beta - (\alpha+\beta)^2}{4}$ (completing the square)

If I add $-\left(\frac{4\alpha\beta - (\alpha+\beta)^2}{4}\right)$, then I get $y = (x - \frac{\alpha+\beta}{2})^2$ which has equal roots at $x = \frac{\alpha+\beta}{2}$.

If you as the reader substitute $\alpha = 1$ and $\beta = -3$, you will get the values I used in the example. The mathematics here is not sophisticated and no doubt many people may know this already. As the reader, you may not have thought about the links before but you may have followed the above explanation easily and said to yourself "oh, obviously!"

In the remainder of the paper I wish to analyse the incident in more detail. The discovery itself is unimportant but I hope that by reflecting on the incident, by opening up my own thinking to the reader, and by bringing theory to bear on practice, I might share something with the mathematics education community that counts far more than quadratic equations. In doing this, I shall consider the following questions:

Why had I not thought about this before?
What enabled me to ask the question?
What enabled me to answer the question?
How can I capitalise on this incident in my work as a mathematics teacher educator?

In dealing with each of these questions, I draw on the five strands of mathematical proficiency (Kilpatrick et al, 2001). The five strands are: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. I describe each strand briefly before proceeding.

The five strands of mathematical proficiency

Conceptual understanding is characterised by an ability to connect individual facts and procedures into an integrated whole, to represent mathematical ideas in multiple ways, and to connect new knowledge into an existing web of concepts. Learners with conceptual understanding know why a mathematical idea is important and when it is useful. Procedural fluency refers to the ability to execute procedures and algorithms accurately, efficiently and in the appropriate context. Strategic competence concerns the ability to formulate, represent and solve mathematical problems. As learners develop their problem solving skills, they are able to use more efficient and sophisticated methods and to see similarities in the structures of apparently different problems. Adaptive reasoning refers to the ability to think logically about a mathematical problem. It includes formal proof as well as informal explanation and justification through pattern, analogy and metaphor. Productive disposition focuses primarily on the learner's beliefs about and attitudes toward mathematics. Learners with a productive disposition see mathematics as useful and logical, and believe that there is value in working hard at maths. Kilpatrick et al (2001) argue strongly that all five strands are intertwined and complement each other.

In our work in pre–service mathematics teacher education we draw heavily on the ideas encapsulated in the five strands. I illustrate this in answering the questions posed above.

Why had I not thought about this before?

It concerned me that I had never thought before about the links between the different forms of the quadratic equation. It seemed such an obvious question to ask and yet I had never considered it, and I don't remember seeing it in text books I have used in my teaching – it may have been there and I just didn't pay it sufficient attention. Perhaps I had not thought about it before because our study of quadratics, particularly in Grade 11, is often disjoint with the solving of quadratic equations treated separately from both curve sketching and nature of roots. Also we have tended to treat graphs as static entities because in the past we have done very little work on transformations of functions at FET level. Another possible reason is that the school curriculum has not encouraged us to explore links within and across different sections of the maths syllabus. These blame–it–on–the–curriculum reasons do not reflect the productive disposition we want teachers and learners to develop. And while I must admit that I also asked myself "why didn't anyone show me this?", I want to take responsibility for my own mathematics learning and be a proactive learner. This leads me to the next question.

What enabled me to ask the question?

One of the key features of our work with pre–service maths teachers is that we encourage students to investigate mathematical ideas more fully than they may have done at school. In some cases this means looking at old (i.e. school) maths in new ways; in other cases it involves exploring new mathematical content. Whether the content is familiar or not, we want students to develop a deep understanding of the mathematics, and this depth is largely dependent on their ability to make and explore links between different aspects of the mathematics they study. We want students to become "question–posers" and not just "question–answerers". We believe that the inclination to pose mathematical questions is characteristic of a productive disposition and we also believe that question–posing, in the mathematical sense (not just the pedagogical sense), is a critical part of the mathematical work that teachers must do. So it is important that prospective teachers learn to do it regularly and successfully. So, I would like to think that what I did at my desk that night reflects that I am "walking the talk".

However, I think there is more to my posing of the question than a productive disposition. I was sitting in front of a computer displaying a graphical representation of a quadratic function and I knew that I could easily get the computer to move the graph. The computer technology provided a tool for me to explore and think about the mathematical idea. Without this visual representation and the ability to manipulate it, I might not have been led to pose my question.

What enabled me to answer the question?

There are several factors that enabled me to answer the question. My conceptual understanding of this area of mathematics enabled me to understand the impact of the parameters (a, b, c, p, q, α and β) on the graph of the function, and to predict how changes in these values would impact the graph. I was able to see the q–value in two ways: as the y–value of the turning point and as a vertical translation of the parabola. This flexibility was key in realising that I needed to make the value of q zero in order to get equal roots. This in turn meant that I needed to translate the parabola vertically through a distance given by the additive inverse of q. This would mean that $p = \alpha = \beta$. I was also able to shift between algebraic and graphical representations of the situation and this was critical to what I had noticed.

My procedural competence enabled me to factorise the quadratic, simplify the algebraic expression, substitute values of x, and complete the square. Being able to do this accurately and efficiently enabled me to maintain focus on the original question and not be side–tracked by algebraic manipulation.

With regard to strategic competence, a key issue was knowing when to use the computer and when to use algebra. My question was posed in an algebraic setting since I was interested in the relationship between the different forms of the equation. My personal preference, however, is to work visually so I shifted very quickly to a graphical setting in order to explore my question further on GSP. The graphical setting and the computer technology provided the means to draw several graphs easily which enabled me to maintain the focus on my question and not get bogged down in point plotting and curve sketching. Having seen the graphs, I then needed to shift back to algebra to investigate how the translations could be represented algebraically and how this would impact on the form of the equation.

Concerning adaptive reasoning: the use of multiple representations enabled me to reason more easily about what I was noticing. I was also able to refer to what I already knew about salient aspects of the quadratic function such as the axis of symmetry and minimum values, and this helped me confirm that my conclusions made sense. The next step was to find a way to communicate this clearly to others which leads me to the next question.

How can I capitalise on this incident in my work as a mathematics teacher educator?

I wanted to be able to use this incident with our pre–service students but I wasn't sure how best to do this. After several days, I decided that there were five aspects that I wanted our students to glean from my experience: to learn *what* I had found from my explorations, to know *how* I had proceeded in my investigation, to gain insight into my thinking as I attempted to transform my new knowledge into a learning experience for others, to gain insight into my decisions about how best to communicate my findings, to see evidence of a teacher continuing to learn mathematics by making new connections within "old mathematics". In addition, I felt that the incident provided a useful opportunity to show how the five strands of mathematical proficiency might be used to analyse a small piece of mathematical work. I started to think about communicating my findings, an aspect that links closely to transforming my new knowledge into a learning experience for others.

I began by writing notes to myself to record my findings concisely. I also made a printout of two graphs I had used. The discussion of my findings at the beginning of this paper is a slight adaptation of those rough notes. The example was chosen carefully to ensure that the roots and the coordinates of the turning point all had integer values since I didn't want messy fractions or irrationals to add unnecessary "noise". My exploration that night had started with $y = 3(x - 1)(x + 2)$ which has an axis of symmetry at $x = -\frac{1}{2}$ and a minimum value at $y = -6.75$. I felt that the added complexity of the fractions (i.e. noise) did not add to an understanding of the issue and might actually detract from the focus, particularly when completing the square. I had a longer debate with myself about the choice of the a–value. While I wanted to choose values that kept the manipulation simple, I didn't want to choose a special case that might lead to confusion at a later stage. My experience with quadratic functions and expressions is that $a = 1$ often simplifies matters too much and hides key issues – think about learning to factorise trinomials when the coefficient of x^2 always has a value of one. Eventually I decided to keep $a = 1$. The key issue is that the a–value doesn't change since the shape of the parabola is not changing. Choosing $a = -1$ or some other value may have added a layer of complexity that wasn't essential to grasp the key ideas I was trying to communicate. I feel strongly, however, that a second example should make use of an a–value that is not one. This would force the reader to consider the impact of the a–value when completing the square and to think explicitly about its impact on the equation of the new graph (albeit that it doesn't change).

Conclusion

I began this paper by sharing a finding about links between different forms of the quadratic equation. The finding itself is not profound and will make no contribution to the existing body of mathematical knowledge. But I hope that you as the reader will have gained from reading about my reflections and journey if not from my mathematical "discovery". And I trust that you will be also encouraged to reflect on your work as a mathematics teacher, in the broadest sense of the word, and then to ask some mathematical question, to explore it and then to share it with the mathematics education community.

References

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