

Number Patterns, Cautionary Tales and Finite Differences

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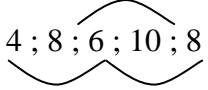
Number Patterns

I recently included the following question in a scholarship examination for Grade 7 pupils:

Consider the following number pattern in which only the first two numbers have been given: 4 ; 8 ; ___ ; ___ ; ___ ; ...

Create three different number patterns (each starting with 4 ; 8) by writing out the next three numbers. In each case explain the rule that controls your number pattern.

I was most pleasantly surprised not only by the diversity and originality of the responses, but in many instances by the clear articulation of the accompanying explanation. Most pupils included the two “obvious” patterns (4 ; 8 ; 12 ; 16 ; 20 and 4 ; 8 ; 16 ; 32 ; 64) but the third pattern led to much creativity. I include the following by way of example:

- Thomas: 4 ; 8 ; 24 ; 96 ; 480
 “× one number by 2 the next by 3 the next by 4 and the next by 5”
- Benjamin: 4 ; 8 ; 14 ; 22 ; 32
 “The number you are adding is going up in twos”
- Craig: 4 ; 8 ; 13 ; 19 ; 26
 “The number you add from 4 to 8 is 4 so as it carries on you make the adding number 1 higher. 4, 5, 6, 7”
- Lochlan: 4 ; 8 ; 4 ; 8 ; 4
 “Plus 4 ; ÷ 2”
- Jordan: 4 ; 8 ; 24 ; 48 ; 144
 “Always double then multiply by 3”
- Kopano: 4 ; 8 ; 15 ; 20 ; 30
 “ $n \times 4 ; n \times 4 ; n \times 5 ; n \times 5 ; n \times 6$ ”
- Gary: 4 ; 8 ; 12 ; 8 ; 4
 “The number pattern starts at 4 and goes up in multiples of four until it reaches 12 and then goes down in multiples of 4 and reaches 4 and carries on like that”
- Matthew: 4 ; 8 ; 10 ; 11 ; 11½
 “½ the number you plussed”
- Ryan: 4 ; 8 ; 6 ; 10 ; 8

 “Both numbers are going up in 2’s in a sequence.”
- Adrian: 4 ; 8 ; 12 ; 20 ; 32
 “Added numbers together (eg. $4 + 8 = 12$)($8 + 12 = 20$)($12 + 20 = 32$)”

Cautionary Tales

In the above examples only the first two numbers of each pattern were given. This necessarily, and indeed purposefully, allowed for great scope of creativity. Let us now limit the degree of freedom somewhat and consider the following sequence: 2 ; 3 ; 5 ; 8 ; ___ ; ...

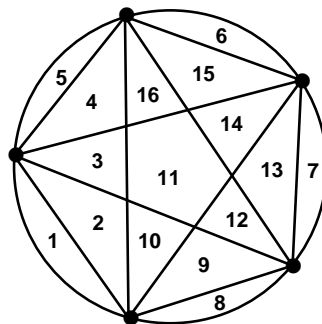
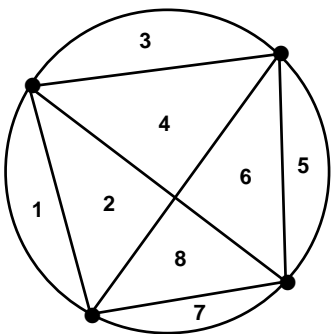
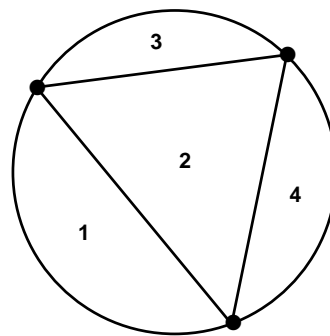
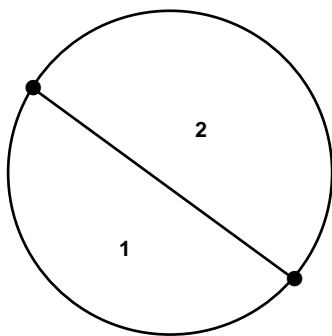
Asked to determine the next number in the above sequence, most pupils would probably respond with an answer of 12 by observing that the difference between consecutive terms increases by 1 each time. But is this the only possible sequence? Some pupils may have recognised that beginning with 5, each term is obtained by adding the two terms immediately preceding it. Following this alternative pattern, the 5th term would be 5 + 8 = 13. But of course there are still other possibilities. Perhaps the sequence is nothing other than a continuous repetition of the first four terms: 2 ; 3 ; 5 ; 8 ; 2 ; 3 ; 5 ; 8 ; 2 ; 3 ; 5 ; 8 ; etc. In such a case the 5th term would be 2. And of course there are countless other possibilities.

A superb example of the necessity for caution and circumspection when looking for patterns is the following well-known scenario:

Investigate the relationship between the number of dots on the circumference of a circle and the number of regions created when joining the dots.

For 1 dot there is obviously only 1 region since the circle is still whole. The scenarios for 2, 3, 4 and 5 dots are shown below. The investigation thus far is summarised in the following table:

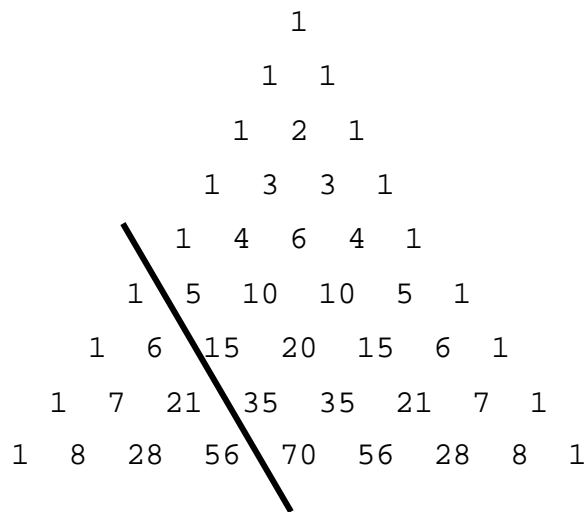
Number of dots	1	2	3	4	5
Number of regions	1	2	4	8	16



One may well be excused for expecting 32 regions for 6 dots, but of course that isn't the answer! For 6 dots the number of regions created is only 31. The following table summarises the situation for up to 8 dots:

Number of dots	1	2	3	4	5	6	7	8
Number of regions	1	2	4	8	16	31	57	99

It is also interesting to note the beautiful connection between the above scenario and Pascal's triangle:



The horizontal **sums** of the rows of Pascal's triangle to the right of the bold line are none other than: 1 ; 2 ; 4 ; 8 ; 16 ; 31 ; 57 ; 99 ; . . . Although the exact position of the bold line may seem somewhat arbitrary, it is nonetheless a comforting and satisfying observation that such a number sequence should appear in a different, and seemingly unrelated, mathematical context. It is left to the reader to investigate this particular connection further.

The Calculus of Finite Differences

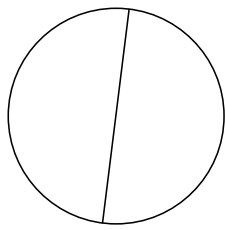
The calculus of finite differences is a useful intermediate step en route from algebra to calculus. At an elementary level it contains a number of procedures that are enormously useful when attempting to determine the general formula for a sequence of numbers.

Let us consider the following scenario:

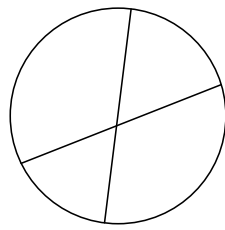
Investigate the *maximum* number of pieces into which a pizza can be sliced by n straight cuts.

In order to find a general formula in terms of n , let us look at the first few cases. If the general formula is not too complex then the method of differences may help us to find it somewhat empirically.

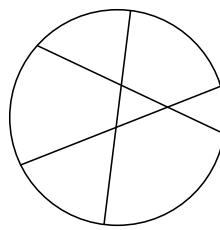
Zero cuts obviously leaves 1 piece. 1 cut produces 2 pieces. 2 cuts produces 4 pieces, etc.



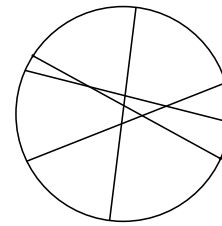
1 cut → 2 pieces



2 cuts → 4 pieces



3 cuts → 7 pieces



4 cuts → 11 pieces

The following table summarises the situation for up to 5 cuts. The rows of differences represent the differences between adjacent terms in the row directly above it.

Number of cuts:	0	1	2	3	4	5
Number of pieces:	1	2	4	7	11	16
First difference:		1	2	3	4	5
Second difference:			1	1	1	1

If the original series is generated by a linear function then the first set of differences will be constant. If the function is quadratic then the second set of differences will be constant. A cubic function will have a constant set of third differences, and so on. In the above example the second set of differences are all the same value, the defining function is thus quadratic. Since we now know that the general formula is of the form $y = an^2 + bn + c$ we can easily solve for a , b and c by substituting three different points e.g. (0;1), (1;2) & (2;4) and solving simultaneously. The general formula thus works out to be: $\frac{1}{2}n^2 + \frac{1}{2}n + 1$

Let us now briefly re-visit the relationship between the number of **dots** (n) on the circumference of a circle and the number of regions created when joining the dots. Let us assume we have diagrammatically confirmed the number of regions for up to 6 dots:

Number of dots	0	1	2	3	4	5	6	7	8
Number of regions	1	1	2	4	8	16	31		
First difference		0	1	2	4	8	15		
Second difference			1	1	2	4	7		
Third difference				0	1	2	3		
Fourth difference					1	1	1		

Since the fourth set of differences is constant, we have established that the defining function is a fourth power expression in n . Rather than determining the actual formula for the general term, we can simply continue the patterns in the rows of differences and hence work backwards to continue the pattern in the “number of regions” row. Since the fourth set of differences are each 1, the third difference sequence must continue with 4 and then 5. Using a similar approach we can continue the sequence of second differences, first differences and ultimately the original sequence:

Number of dots	0	1	2	3	4	5	6	7	8
Number of regions	1	1	2	4	8	16	31	57	99
First difference		0	1	2	4	8	15	26	42
Second difference			1	1	2	4	7	11	16
Third difference				0	1	2	3	4	5
Fourth difference					1	1	1	1	1

Rather than solving 5 equations simultaneously to determine a formula for the general term, we can make use of Newton's formula:

$$a + bn + \frac{cn(n-1)}{2} + \frac{dn(n-1)(n-2)}{2 \times 3} + \frac{en(n-1)(n-2)(n-3)}{2 \times 3 \times 4} + \dots$$

The formula is applicable to all cases, regardless of the degree of the defining function. However, it assumes that the series begins with the value of the function when n is zero. This number is a . The first number in the first row of differences is b , the first number in the second row of differences is c , and so on. The formula is used up to whichever row of differences is constant. Thereafter all differences would be zero and consequently all further additions to the formula would be zero. In our particular case we arrive at the following general formula:

$$\frac{n^4 - 6n^3 + 23n^2 - 18n + 24}{24}$$

Newton's formula applied to quadratic equations yields a useful little trick which is well worth remembering.:

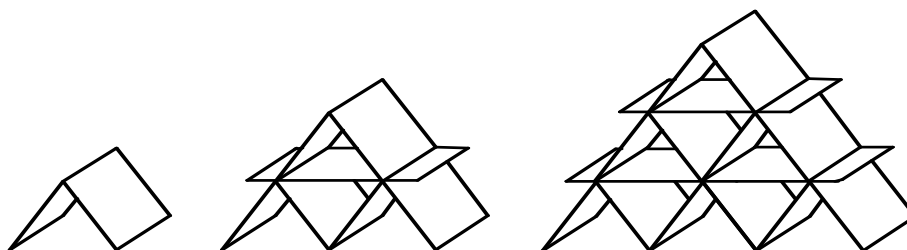
Let us take the function $y = 3x^2 - 2x + 5$ by way of example. Firstly substitute for $x = 0$, $x = 1$ and $x = 2$ in order to determine the three corresponding y values viz. 5, 6 and 13. Next work out the first and second rows of differences:

$$\begin{array}{ccc} 5 & 6 & 13 \\ & 1 & 7 \\ & & 6 \end{array}$$

The coefficient of x^2 is always half the bottom number. The coefficient of x is obtained by subtracting half the bottom number from the first number of the middle row. The constant in the formula is simply the first number of the top row.

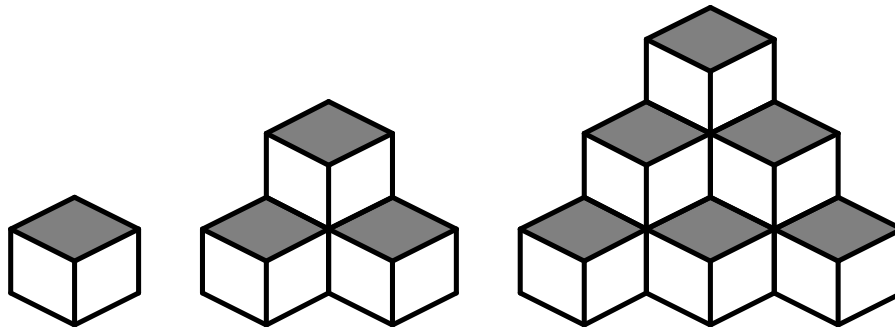
Here are a few investigations for you to explore:

1. The maximum number of electrons that can occupy each "shell" of an atom creates the following sequence as we move outward from the nucleus: 0 ; 2 ; 8 ; 18 ; 32 ; 50. Determine the general formula.
2. Investigate the relationship between the number of dots on the circumference of a circle and the number of lines needed to join each dot to every other dot. Alternatively, if each person at a meeting shakes hands once with every other person, determine an expression for the number of handshakes there would be if n people were present at the meeting. These two problems are essentially equivalent.
3. Consider building tiers of cards as follows:



Determine a formula to work out the total number of cards you will need for any given number of levels.

4. Imagine the following structures built up using solid cubes:



Determine the general formula for the number of cubes needed as a function of the number of levels in the structure.

Solutions:	1.	$2n^2$	3.	$\frac{3}{2}n^2 + \frac{1}{2}n$
	2.	$\frac{1}{2}n^2 - \frac{1}{2}n$	4.	$\frac{1}{6}n(n+1)(n+2)$

Newton’s formula assumes that the series begins with the value of the function when n is zero. This zero value can often cause an upset to an otherwise relatively simple pattern. Consider the case of dividing a plane up into different parts by intersecting circles of the same size. Clearly with no circle there is 1 part. However, the general formula for such a scenario is $n^2 - n + 2$ which would seem to suggest 2 parts for zero circles, which of course is somewhat meaningless. Where the zero case causes an upset to the pattern, the easiest approach is to extrapolate backwards in the rows of differences and assume a suitable value for the zero case that produces the desired first number in the last row of differences.

On a final note of caution, it is worth pointing out that for a finite sequence of numbers there is an infinite number of functions that could generate the sequence. This is equivalent to plotting a finite number of points in the Cartesian Plane where there would obviously be an infinity of curves that one could draw through the given points. By using the method of differences all we have determined is the lowest order curve that can be drawn through the given points. There is no guarantee that further observations will not require a modification of the general formula, although there are usually independent approaches one can use to prove a “guessed” formula.

In the words of George Polya: “Nature may answer Yes or No, but it whispers one answer and thunders the other. Its Yes is provisional, its No is definitive.”

References

Bolt, B. (1984) *The Amazing Mathematical Amusement Arcade*, Cambridge: Cambridge University Press.

Gardner, M. (1966) *Martin Gardner’s New Mathematical Diversions From Scientific American*, London: George Allen and Unwin Ltd., pp. 234 – 246.

Posamentier, A. S. and Hauptman, H. A. (2001) *101 Great Ideas for Introducing Key Concepts in Mathematics – A Resource for Secondary School Teachers*, Thousand Oaks, Ca.: Corwin Press Inc., pp. 28 – 30.

Sawyer, W. W. (1943) *Mathematician’s Delight*, Middlesex: Penguin, pp. 90 – 100.