Do Infinite Things Always Behave Like Finite Ones?

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In a calculus class for pre-service teachers the lecturer asked the prospective teachers to explore whether 0.999... is equal to 1. Unsurprisingly the majority of the class's initial response was that 0.999... is less than 1. After some research and through discussion with the lecturer some groups of students came up with arguments that show that 0.999... = 1. The following four of those arguments were presented to the class:

Argument 1:
Let $x = 0.999....$
Then $10x = 9.999...$
Subtracting the first equation from the second gives: $9x = 9$, and so $0.999\ldots = x = 1$.

Argument 2:
We know that $0.333... = \frac{1}{3}$.
If we multiply both sides of this equation by 3 we get 0.999... = 1.

Argument 3:
$0.999\ldots = 0.9 + 0.09 + 0.009 + \cdots$.
$0.9 + 0.09 + 0.009 + \cdots$ is a geometric series with first term $a = 0.9$, common ratio $r = 0.1$ and so using the formula for the sum to infinity of a geometric series we get:
$$S_\infty = \frac{a}{1-r} = \frac{0.9}{1-0.1} = \frac{0.9}{0.9} = 1.$$
So $0.999\ldots = 0.9 + 0.09 + 0.009 + \cdots = 1$.

Argument 4:
$\frac{1}{9} = 0.111\ldots$, $\frac{2}{9} = 0.222\ldots$, and so on, until we get to $\frac{8}{9} = 0.888\ldots$ and finally to $1 = \frac{9}{9} = 0.999\ldots$.

After a few questions and with some clarification the students in the class accepted these arguments. At a later stage the students were asked to write about 0.999... in their journals. One student's comment perhaps sums up best what the majority of students in the class expressed: "Although we have proved 0.999...=1, I still feel that 0.999... is actually just a little less than 1."

Now it is well documented that students (of all ages and at all stages of mathematical learning) have difficulty with the notion that 0.999... = 1 and with the arguments presented above. When we (two mathematicians and a mathematics educator) started looking at the mathematical ideas involved in the arguments we started to see layers of "big" ideas and complexity in this one little example. For example, the first, second and fourth argument rely on the idea that multiplication works in the ordinary way with an infinite decimal and the third argument relies on "the formula" for an infinite sum. In all cases we are working with infinite things without really questioning whether infinite things really can just be played with in the same way finite things can. In a sense if you look at each of these arguments above we are asking students to do the kind of arithmetic they would always do with "ordinary" finite decimal numbers, but to accept at the same time something mysterious (i.e something that looks so clearly to be less than 1 is in fact equal to 1) that can really only happen when infinity comes into play.

We found that if one of us played the sceptic and took nothing in the arguments for granted this opened the layers of mathematical ideas for scrutiny. We are not suggesting that what is discussed here is necessarily suitable for the school classroom, but we feel the understanding we generated through our discussion provided insights into aspects of the school curriculum (infinite recurring decimals, infinite sums and limits). So what we want to share with you here is dialogue between a teacher (T) who plays the role of the sceptic, and a mathematician (M), about the equality 0.999... = 1 and the above "proofs" that this is so.

Act 1 Scene 1: Discovery - Recurring decimals are worrying things!

T: Look, we all agree that 0.99 <1.00, so surely it should also be the case that 0.999… < 1.000… ?

M: But then what are you saying about the proofs we've just seen that say they are equal? What's going on there? Are you saying that there is something wrong with the "proofs" or is there some other possibility?

T: Maybe infinite decimal expansions (I mean the non-terminating ones) don't behave properly? Maybe there is something wrong with them? Or maybe they behave differently from finite or terminating decimal expansions?

M: OK, back to basics. How do we know that 0.99 < 1.00 ? What is the difference between them?

T: Well, if we subtract, then we get

$$\begin{array}{r} {}^{0}\cancel{1}\,.\,{}^{1}\cancel{0}\,{}^{1}0 \\ -\,0\,.\,9\,9 \\ \hline -\;0\,.\,0\,1 \\ \hline \end{array}$$

Since that difference is greater than zero, 1.00 must be larger than 0.99.

M: Take that a bit further then.

T: OK; we try the same with 0.999… and 1.000. We need to do:

$$\begin{array}{r} 1\;.0000000000\ldots \\ -\;0\,.9999999999\ldots \\ \hline \\ \hline \end{array}$$

To do this we start on the right as usual. Oh; I'm in trouble. There isn't any thing at the right to start with since the digits don't stop and so there is a carrying problem.

M: Are you saying that you're not sure how we should subtract infinite decimal expansions?

T: Yes, that's exactly what I think. I don't know how to do the subtraction for unending decimals. So I'm not really so sure now whether 0.999… is obviously smaller that 1.00 or not.

M: What about the argument $1 = 3 \times \frac{1}{3} = 3 \times 0.333\ldots = 0.999\cdots$?

T: Well, I sort of buy it, but now I'm going to get a bit picky (just like you mathematicians). This argument *does* involve multiplying an infinite decimal expansion by 3, and it doesn't seem to present any problems here. But I was just wondering: what about a multiplication like

$4 \times 0.3333\ldots$? Where do we start the multiplication? I think this is going to be just as difficult as handling $1.0000\ldots - 0.999\ldots$ .If we don't have a way of handling this multiplication, what does this have to say about how much we can rely on our previous multiplication?

M: Are you saying something about your ideas of multiplication? Try to formulate these.

T: I'm saying, I think, that I no longer trust multiplication of infinite decimals.

M: Can you think of other situations where you might have the same worries? Some other operations perhaps?

T: Maybe there are problems with addition as well? Something like $0.222\ldots + 0.333\ldots = 0.555\ldots$ seems unproblematic. But what do we do with $0.333\ldots + 0.777\ldots$? Does this mean that we cannot do arithmetic with infinite decimal expansions? And that we can no longer believe two of the "proofs" that $0.999\ldots = 1.000$ that we started with? There is something distinctly fishy about all of this.

M: Why?

T: All the examples above used infinite recurring decimal expansions, and surely these represent rational numbers. And we do know that we can do arithmetic with rational numbers! We teach about adding and multiplying fractions quite early on in school. So what's wrong with decimals?

M: Let's examine the argument $1 = 3 \times \frac{1}{3} = 3 \times 0.333\ldots = 0.999\cdots$ once more. There can be no argument about the first equality. The second one relies on the "fact" that $\frac{1}{3} = 0.333\cdots$. Why can we say this?

T: That sounds like a really pedantic question. Surely one simply divides 1 by 3; it goes in 0 times and the remainder is 1. Hence the first digit in the expansion is 0. Then you divide 10 (= 10 times the remainder) by 3; it goes in three times, and the remainder is 1. This gives 3 as the first digit after the decimal point. The other 3's follow in the same way, without end, because there always is a remainder of 1.

M: Granted, this gives a *method* for obtaining a decimal expansion of one third. But why does it work? How does it follow that we have the *equality* $\frac{1}{3} = 0.333\cdots$? Wouldn't you now be tempted, just as you are with 0.999... and 1.0, to say that $0.333\ldots$ is actually just a bit less than $\frac{1}{3}$? Why do you believe that $0.333\ldots$ really is exactly $\frac{1}{3}$?

T: Oh come on; everyone knows that; we learned that back in grade 6.

M: But it seems to me that we can only make sense of the equation $\frac{1}{3} = 0.333\cdots$ if we are clear about what both sides of the equation mean. Let's accept that we understand $\frac{1}{3}$. Now what about the right hand side of the equation? You can't rely solely on the fact that you get $0.333\ldots$ when you do the long division of $1 \div 3$ to give meaning to the recurring decimal notation because there isn't any long division that is going to give us $0.999\ldots$ or $0.4999\ldots$, for example. I'll try to make myself a bit clearer.

T: That would be nice!

Act 1 Scene 2: Trying to understand recurring decimals - having to add up infinitely many things!

M: Let's start with something simpler: what does 0.3 stand for?

T: That is simply shorthand for $\frac{3}{10}$.

M: OK, take this example further. What about 0.33?

T: Oh, I see where you're going. 0.33 is shorthand for $\frac{3}{10}+\frac{3}{100}$. Then 0.333… must be shorthand for $\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\cdots$. Oh dear; there are those dots again. We haven't made any progress. Now you want me to add *infinitely many* terms? Surely this is not possible!

M: OK, so let's admit that at this stage we are *not* clear what 0.333… means. But then if we admit that it is going to be tricky to discuss the equation $\frac{1}{3}=0.333\cdots$. We really have a problem. Let's forget about this equation for the moment and imagine that we have complete freedom to give our own meaning to 0.333… . Rather than make an arbitrary choice, we would like to make a sensible decision that would work for *any* infinite decimal expansion and would fit in with what we already know about the meaning of finite decimal expansions.

T: If that is possible!

M: Let's take the last part of the previous sentence as our starting point: we can make sense of the finite decimal expansions 0.3, 0.33, 0.333, and so on. To cut down on the writing, let's write

$$S_1=\frac{3}{10}$$
$$S_2=\frac{3}{10}+\frac{3}{100}$$
$$S_3=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}$$

and so on; in general, if n is a positive integer, we write

$$S_n=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\cdots+\frac{3}{10^n}.$$

This means that S_n is the sum (that's why we used S) of the first n terms in the infinite sum that we are trying to make sense of. Speaking technically, such sums are usually called *partial sums.* If you look at the formula for S_n, what do you notice?

T: Oh yes, I can see that S_n is a geometric series with n terms, first term $\frac{3}{10}$ and common ratio $\frac{1}{10}$. We've got a formula for that: it is

$$S_n=\frac{\frac{3}{10}(1-\frac{1}{10^n})}{1-\frac{1}{10}}=\frac{1}{3}(1-\frac{1}{10^n})=\frac{1}{3}-\frac{1}{3}\frac{1}{10^n}.$$

M: There's a lot you can actually see from your formula. From it we can make three interesting observations. Here's the first one:

For every n, $S_n<\frac{1}{3}$;

T: That's because $1-\frac{1}{10^n}$ is always less than 1, no matter what *n* is.

M: Good; here's the second observation:

$$\text{For every } n, \quad \frac{1}{3} - S_n = \frac{1}{3}\frac{1}{10^n}.$$

T: Oh, I think I can get that by subtracting S_n from $\frac{1}{3}$. And this form helps me see that as *n* gets bigger the difference between $\frac{1}{3}$ and S_n decreases.

M: Indeed. The final observation:

$$\text{For every } n, \quad S_n < S_{n+1}.$$

T: So although the finite sums S_n are always strictly less than $\frac{1}{3}$, the sums get larger and closer and closer to $\frac{1}{3}$ as *n* increases .

M: Recall one of the things we're trying to do with all this stuff is to create a meaning – a definition if you like - for 0.333… We want this to make sense to us, but also be generalisable to other infinite recurring decimals.

Now we've just seen above that the finite sums are getting closer and closer to something. So what if we declare 0.333… to be the something the finite sums get closer and closer to?

T: OK I agree that declaring 0.333…to the something the finite sums get closer and closer to does make sense – and we could apply a similar idea to other infinite recurring decimals.

M: Right – this is an important step. We can think of our declaration above as a definition of what we mean by an infinite sum: it is the thing (if such a thing exists) that the finite sums get closer and closer to. So next we need to ask ourselves what *is that something* that our finite sums are getting closer and closer to. The observations above suggests very strongly that that something is $\frac{1}{3}$. And so we're saying that 0.333… is really $\frac{1}{3}$.

T: No, I don't buy that at all; it's like you've just defined the problem away.

M: I see what you mean, but it really is a two step procedure. We decided that
(a) 0.333… ought to stand for whatever the numbers given by the partial sums were getting closer and closer to. (That was indeed a definition!)

After deciding that, we noticed that
(b) they really are getting closer and closer to $\frac{1}{3}$.

Thus we were forced to decide that, according to our plan, 0.333… has to be $\frac{1}{3}$.

T: I can go along with your decision that 0.333… is that something to which the finite sums are getting closer and closer to, but I still do not believe it is $\frac{1}{3}$, I think it is something *just* smaller, for all the sums are strictly less than $\frac{1}{3}$.

Act 1 Scene 3: We ask why 0.333... can't be just a teeny bit smaller than $\frac{1}{3}$.

M: OK; let's try to see what we can say. We'll examine your claim more closely. Let's write x for your something, and list some properties it seems to have. I hope I can convince you that your x is going to cause us major difficulties; in fact it is going to lead us to a contradiction! Here are the properties:

1) For every n, $S_n < x$.
2) As n increases, S_n gets closer and closer to x.
3) Now you also want $x < \frac{1}{3}$.

Now 3) means that $\frac{1}{3} - x$ is a *positive* number; very small perhaps, but still larger than 0. We'll see that this assumption leads us into real difficulties! Now you know that the number $\frac{1}{10^n}$ becomes smaller and smaller as n becomes larger; so we can make it as small as we like by simply choosing n large enough.

T: True enough

M: This means that it will be possible to choose n so large that $0 < \frac{1}{10^n} < \frac{1}{3} - x$.

And we know that $0 < \frac{1}{3 \times 10^n} < \frac{1}{10^n}$ and so we have that $0 < \frac{1}{3}\frac{1}{10^n} < \frac{1}{3} - x$

T: That is we have that $\frac{1}{3}\frac{1}{10^n}$ is smaller than our difference between x and $\frac{1}{3}$.

M: Yes. Now, using the fact that $\frac{1}{3} - S_n = \frac{1}{3}\frac{1}{10^n}$ (which we worked out much earlier), we get $0 < \frac{1}{3} - S_n < \frac{1}{3} - x$, but if $\frac{1}{3} - S_n < \frac{1}{3} - x$, it must follow (do the rearranging yourself) that $S_n > x$. But that **can't be**, because we know that $S_n < x$. There's our contradiction!

T: So now you're going to claim that there *cannot be* an x smaller than $\frac{1}{3}$ to which the finite sums get closer and closer. We therefore appear to be justified in saying that $\frac{1}{3}$ is *the something* to which the finite sums are getting closer and closer to.

M: I think it's important to note here that what made it possible for us to meet your objection was the fact that the distance between $\frac{1}{3}$ and S_n is equal to $\frac{1}{3}\frac{1}{10^n}$, and that we can make $\frac{1}{3}\frac{1}{10^n}$ as small as we like by making n large enough. So that allows us to find something even smaller than the difference between x and $\frac{1}{3}$. It's rather cunning.

Act 1 Scene 4: Using the same approach to settle the 0.999... problem

M: Can we get back to the more worrying problem? This is a build up for the original problem of making sense of 0.999… Let's try our new approach there. We have managed to attach a meaning to one infinite decimal expansion. Can we do it in the same way for other such expansions?

T: If we use the same approach on 0.999… , we get

$$S_n = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots + \frac{9}{10^n}$$

and

$$S_n = \frac{\frac{9}{10}(1 - \frac{1}{10^n})}{1 - \frac{1}{10}} = (1 - \frac{1}{10^n}).$$

M: OK, so what happens now for larger and larger *n*?

T: Oh yes; the same argument as before then shows that it makes sense to say that 0.999… = 1 since the difference between the finite sums and 1 really disappears as *n* gets bigger and bigger.

M: Well, "really disappears" may be pushing it, but it certainly gets smaller and smaller, so that the finite sums get closer and closer to 1.

T: And using a similar argument as in the 0.333... example, I can see that we can also show that 0.999... *can't be anything just a bit less than 1!*

It is perhaps worth reflecting now on what the problem with the equality 0.999…= 1 was, and how it was solved. To understand the equation, we needed to understand what the left-hand side meant. Saying that it stood for a sum of infinitely many numbers highlighted the problem, rather than solving it. It is impossible to add infinitely many numbers; we do not have the time! This means that we had to give a meaning to such a sum – we had to *define* what we mean by it. In the dialogue we tried to show how this can be done in a way that makes sense, that we feel comfortable with. Once we had decided on a definition, we had to stick to meaning assigned by this definition, and live with any result that we derived using this definition (such as 0.999… = 1, for example).

At this stage it looks as if we have sorted out the problem of infinite decimals expansions and how to deal with them. We are suggesting that a sensible way to handle infinite decimals is to think of them as infinite sums, and that this leads to a consistent way of handling them and this is indeed the view of most mathematicians these days.

But have we really settled the problem? A sceptic may well say, with some justification, that we have replaced the problem of dealing with the infinite sums that seem to arise in the processes above with a new problem. We have said above that an infinite sum exists if the partial sums get closer and closer to one specific number, which we then call the sum. But what exactly do we mean by "get closer and closer to a number"? How close do we have to get? And how soon? How do we determine when a number qualifies as the sum? We hope to explore some of these questions in a subsequent dialogue. This will be Act 2!