Let Me Count The Ways

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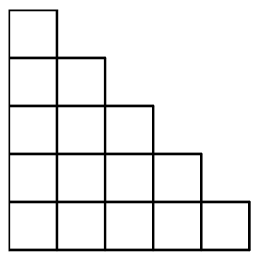
At the Teachers Without Borders Workshop held at Trinset, Mthatha during the winter break, educators were presented the problem of finding the sum, 1 + 2 + 3 + 4 + 5 +… *n*, by finding the area of a staircase figure that has a base of *n* squares.



The area of the staircase is $\frac{n(n+1)}{2}$, but there are a variety of ways to determine this expression. Educators were asked to work in groups and discuss how to solve the problem. A variety of methods emerged, some of which were new to the facilitator. Following is a report of the findings.

Using a Triangle

In this method, a line is drawn to create a triangle. The line could be drawn in more than one way. Each of the methods below refers to the shapes shown in Figure 1.

A. The area of the staircase is the area of the triangle plus the sum of the areas of the half-squares: $\frac{1}{2}(n)(n)+\frac{1}{2}n$.

B. The area of the staircase is the area of the triangle minus the sum of the areas of the half-squares: $\frac{1}{2}(n+1)(n+1)-\frac{1}{2}(n+1)$.

C. The area of the staircase is the same as the area of the triangle: of course one needs to be convinced that the area in the staircase that is not in the triangle is the same as the area that is in the triangle, but not in the staircase. This can be shown to be true and is an interesting exercise.

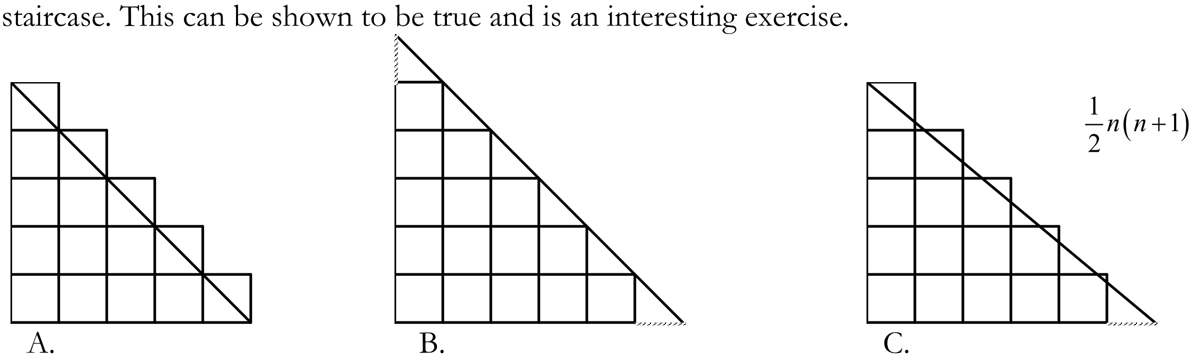


Figure 1

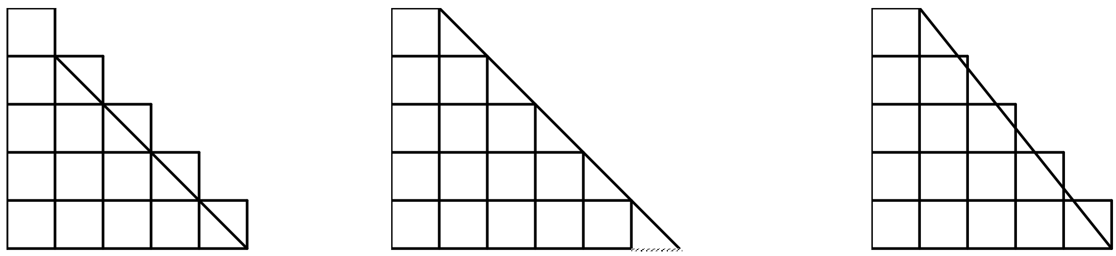
Using a Trapezium

In this method, a line is drawn to create a trapezium. The line could be drawn in more than one way. Each of the methods below refers to the shapes shown in Figure 2.

A. The first term is the area of the trapezium, the second term is the sum of the half-squares, and the final term is the single square at the top: $\frac{1}{2}(n-1)(n+1)+\frac{1}{2}(n-1)+1$

B. The first term is the area of the trapezium and the second term is the area of the extra half-squares at the right that must be excluded: $\frac{1}{2}n\left[1+(n+1)\right]-\frac{1}{2}n$.
Notice that we could also have separated the *n* squares in the column on the left and then used the triangle formula (A) replacing *n* with $n-1$ to find: $\frac{1}{2}(n-1)(n-1)+\frac{1}{2}(n-1)+n$.

C. Unfortunately, as before with the triangle (method C), one needs to be convinced that the area in the staircase that is not in the trapezium is the same as the area that is in the trapezium, but not in the staircase: $\frac{1}{2}n(n+1)$.



A. B. C.

Figure 2

We could also draw the line as shown in Figure 3. The area of the staircase is the area of the trapezium minus the area of the half-squares, plus the area of the top square: $\frac{1}{2}(n-1)\left(2+(n+1)\right)-\frac{1}{2}(n-1)+1$. We could also separate the vertical column on the left and compute the area of the remaining trapezium much as we did in B, and then add the *n* squares in the column: $\frac{1}{2}(n-1)(n+1)-\frac{1}{2}(n-1)+n$.

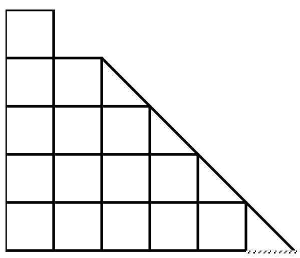


Figure 3

Forming a Rectangle

Another method that some groups used was that of forming a rectangle using a copy of the original staircase, as shown in Figure 4.

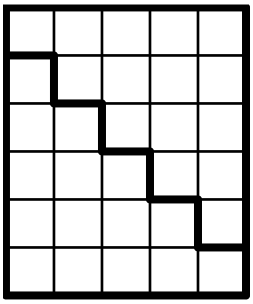


Figure 4

The expression for the area of the staircase with n squares in the bottom is half the area of the rectangle, $\frac{n(n+1)}{2}$.

One group had an interesting variation on this method. They first enclosed the staircase in a square as shown in Figure 5.

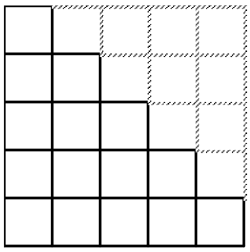


Figure 5

Then they determined that the part they added was one row shorter than the original, so they temporarily separated the bottom row of the original staircase as shown in Figure 6.

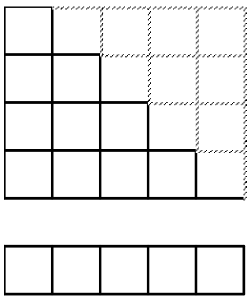


Figure 6

The area of the staircase is half the area of the large rectangle plus the area of the (separated) bottom row of squares, or $\frac{(n)(n-1)}{2}+n$.

Dissect and Rearrange to Form a rectangle.

This method involves taking the squares and forming a rectangle, doing so in a systematic way.
With a staircase with one square we have a 1 × 1 rectangle. With a staircase with 2 squares on the bottom, we can form a 1 × 3 rectangle. With a staircase with 3 squares on the bottom, we can form a 2 × 3 rectangle. With a staircase with 4 squares on the bottom, we can form a 2 × 5 rectangle, as shown in Figure 7.

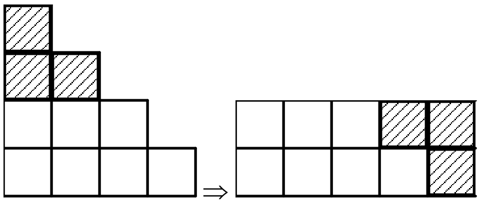


Figure 7

We can continue the process, tabulate the results and start to look for a pattern.

Number of Squares in Bottom Row	Dimensions
1	1 × 1
2	1 × 3
3	2 × 3
4	2 × 5
5	3 × 5
6	3 × 7
7	4 × 7
8	4 × 9

We want to associate the input (Number of Squares in Bottom Row) with the Dimensions.
If n is even, we have dimensions $\frac{n}{2}\times(n+1)$ and if n is odd, we have dimensions $\frac{n+1}{2}\times n$. Each of these expressions can be written as $\frac{n(n+1)}{2}$.

Forming a Square

One group of educators enclosed the staircase in a square as shown in Figure 8.

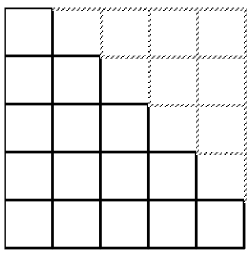


Figure 8

They first wrote 5^2-10. Then they looked at how they obtained 10, and recognized that it was found from a smaller staircase, so they wrote how they would find the area of that staircase, 4^2-6. Continuing, they then wrote 6 as 3^2-3; and finally they wrote 3 as 2^2-1, or 2^2-1^2.

Writing all of this with nested parentheses, we have:

$5^2 - \{4^2 - [3^2 - (2^2 - 1^2)]\}$, and simplifying gives

$5^2 - 4^2 + 3^2 - 2^2 + 1^2$, which rearranges to $5^2 + 3^2 + 1^2 - 4^2 - 2^2$, or $(5^2 + 3^2 + 1^2) - (4^2 + 2^2)$,

which is the sum of the odd squares less than or equal to 5, minus the sum of the even squares less than 5.

In general, if n is odd, then 1 + 2 + 3 + 4 + 5+ … + n is $[1^2 + 3^2 + 5^2 + ... + n^2] - [2^2 + 4^2 + 6^2 + ... + (n-1)^2]$, and if n is even, then 1 + 2 + 3 + 4 + 5 + … + n is $[2^2 + 4^2 + 6^2 + ... + n^2] - [1^2 + 3^2 + 5^2 + ... + (n-1)^2]$.

While not an easier way to compute the sum, it is nonetheless a beautiful pattern.

Additional Investigations

By looking at $5^2 - 4^2 + 3^2 - 2^2 + 1^2$ as $(5^2 - 4^2) + (3^2 - 2^2) + 1^2$ we can see some interesting geometry. Consider the following investigations:

1. What does $5^2 - 4^2$ look like, geometrically? Likewise, what does $3^2 - 2^2$ look like, geometrically? How can you describe, geometrically, the sum $(5^2 - 4^2) + (3^2 - 2^2) + 1^2$?

2. Consider again, $(5^2 - 4^2) + (3^2 - 2^2) + 1^2$, but this time, simplify to $9 + 5 + 1$. Separate the staircase into 3 groups, one with 9 squares, one with 5 squares and a single square. Consider other staircases and look for a pattern.

3. Revisit "**Forming a Square**" and make observations based on investigation 2.

VIA THE WEB

Problem of the Week

Looking for problem solving questions, investigations or brain teasers? For a large set of links to Problem of the Week websites for all ages visit the following website:

http://www.potw.co.za