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Developing Deep Mathematical Thinking through Mathematics Teaching

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Foreword:

Developing deep mathematical thinking is an important aspect to consider when teaching mathematics. The plenaries, presentations and workshops of the AMESA 2019 Congress highlight and explore this aspect further in addition to various approaches to assist teachers in the mathematics classroom. The presentations that are discussed highlight the sharing of ideas, resources and good practice.

The themes of teaching using technology based pedagogies, indigenous knowledge, decolonization, games, problem solving and investigations, provide teachers and teacher educators with interesting avenues to explore to promote deep mathematical thinking when teaching mathematics. Developing deep mathematical thinking through mathematics teaching within Higher Education and through research in mathematics education provides a further glimpse of the possibilities that exist to empower teachers and teacher educators to encourage deep mathematical thinking when teaching mathematics.

We urge delegates to reflect on these ideas, practices and resources as a way of rethinking how mathematics is taught in South Africa. It is only through dedication, commitment, recognizing ones' role and responsibility when teaching as well as actively debating our challenges that we can develop deep mathematical thinking through mathematics teaching. It is through this reflection, sharing and rethinking that we can be proud of teaching and learning mathematics in South Africa.

Academic Coordinator: Jayaluxmi Naidoo

July 2019

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ABSTRACTS

MATHEMATICS AND LANGUAGE: RECOMMENDATIONS FOR MATHEMATICS INSTRUCTION FOR STUDENTS LEARNING THE LANGUAGE OF INSTRUCTION

Judit Moschkovich

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This paper outlines recommendations for mathematics instruction that supports language and is aligned with current research and standards in mathematics education. The recommendations are motivated by a commitment to improving mathematics learning through language for all students and especially for students who are learning the language of instruction (LOI). These recommendations are intended as principles to guide teachers and teacher educators in developing their own approaches to supporting mathematical reasoning, sense making, and language for students who are learning the LOI.

INTRODUCTION

There are multiple uses of the term *language*: to refer to the language used in classrooms, in the home and community, by mathematicians, in textbooks, or on test items. These recommendations for teaching are based on research that often runs counter to commonsense notions of what the term language means. The meaning of language in mathematics teaching should not be reduced to single words and the proper use of grammar (Moschkovich, 2002). Work on the language of specific disciplines provides a complex view of mathematical language (e.g., Pimm, 1987) as not only specialized vocabulary (new words and new meanings for familiar words) but also as extended discourse that includes syntax and organization (Crowhurst, 1994), the mathematics register (Halliday, 1978), and discourse practices (Moschkovich, 2007c). I use a sociolinguistic framework to frame these recommendations and from this theoretical perspective, language is a socio-cultural-historical activity. I use the phrase "the language of mathematics" not to mean a list of vocabulary or technical words but the communicative competence necessary and sufficient for competent participation in mathematical discourse practices[1].

[1] I sometimes use the term "language(s)" as a reminder that there is no pure language and that all language is hybrid.

It is difficult to make generalizations about the instructional needs of all students who are learning the LOI. Specific information about students' previous instructional experiences in mathematics is crucial for understanding how bilingual/multilingual learners communicate in mathematics classrooms. Classroom instruction should be informed by knowledge of students' experiences with mathematics instruction, their language history, and their educational background. In addition to knowing the details of students' experiences, research suggests that high-quality instruction for students learning the LOI that supports student achievement has two general characteristics: a view of language as a resource, rather than a deficiency; and an emphasis on academic achievement, not only on learning the LOI (Gándara and Contreras, 2009).

Research provides general guidelines for instruction for this student population. Students who are learning the language of instruction (LOI) need access to curricula, instruction, and teachers proven to be effective in supporting academic success for this student population. The general characteristics of such environments are that curricula provide "abundant and diverse opportunities for speaking, listening, reading, and writing" and that instruction "encourage students to take risks, construct meaning, and seek reinterpretations of knowledge within compatible social contexts" (Garcia & Gonzalez, 1995, p. 424). Teachers with documented success with students from non-dominant communities also share some characteristics: a) high commitment to students' academic success and to student-home communication, b) high expectations for all students, c) the autonomy to change curriculum and instruction to meet the specific needs of students, and d) a rejection of models of their students as intellectually disadvantaged.

Research on language that is specific to mathematics instruction for this student population provides several guidelines for instructional practices for teaching mathematics. Mathematics instruction for this student population should: 1) treat language as a resource, not a deficit (Gándara and Contreras, 2009; Moschkovich, 2000); 2) address much more than vocabulary and support students' participation in mathematical discussions as they learn the LOI (Moschkovich, 1999, 2002, 2007a, 2007b, 2007d); and 3) draw on multiple resources available in classrooms—such as objects, drawings, graphs, and gestures—as well as home languages and experiences outside of school. This research shows that students, even as they are learning the LOI, can participate in discussions where they grapple with important mathematical content[2]. Instruction for this population should not emphasize low-level language skills over opportunities to actively communicate about mathematical ideas. One of the goals of mathematics instruction should be to support all students, regardless of their proficiency in the LOI, in participating in discussions that focus on important mathematical concepts and reasoning, rather than on pronunciation, vocabulary, or low-level linguistic skills. By learning to recognize how students express

[2] For examples of lessons where English learners participate in mathematical discussions, see Moschkovich, 1999 and Khisty, 1995.

their mathematical ideas as they are learning the LOI, teachers can maintain a focus on mathematical reasoning as well as on language development.

Research also describes how mathematical communication is more than vocabulary. While vocabulary is necessary, it is not sufficient. Learning to communicate mathematically is not merely or primarily a matter of learning vocabulary. During discussions in mathematics classrooms, students are also learning to describe patterns, make generalizations, and use representations to support their claims. The question is not whether students who are learning the LOI should learn vocabulary but rather how instruction can best support students as they learn both vocabulary and mathematics. Vocabulary drill and practice is not the most effective instructional practice for learning either vocabulary or mathematics. Instead, vocabulary and second-language-acquisition experts describe vocabulary acquisition in a first or second language as occurring most successfully in instructional contexts that are language-rich, actively involve students in using language, require both receptive and expressive understanding, and require students to use words in multiple ways over extended periods of time (Blachowicz and Fisher, 2000; Pressley, 2000). In order to develop written and oral communication skills students need to participate in negotiating meaning (Savignon, 1991) and in tasks that require output from students (Swain, 2001). In sum, <u>instruction should provide opportunities for students to actively use mathematical language to communicate about and negotiate meaning for mathematical situations</u>.

The recommendations provided in this paper focus on teaching practices that are simultaneously: a) aligned with current research and standards in mathematics education, b) support students in learning the LOI, and c) support students in learning important mathematical content. Overall, the recommendations address the following questions: How can instruction provide opportunities for mathematical reasoning and sense making for students who are learning the LOI? What instructional strategies support these students' mathematical reasoning and sense making skills? How can instruction help students communicate their reasoning effectively in multiple ways?

ALIGNMENT WITH RESEARCH

Current research in mathematics education (and standards such as the Common Core State Standards or CCSS) provide guidelines for how to teach mathematics for understanding by focusing on students' mathematical reasoning and sense making. Here I summarize four emphases to describe how mathematics instruction for students learning the LOI needs to begin by taking these four areas seriously.

Emphasis #1-Balancing conceptual understanding and procedural fluency: Instruction should balance student activities that address both important conceptual and procedural knowledge related to a mathematical topic and connect the two types of knowledge.

Emphasis #2-Maintaining high cognitive demand: Instruction should use high-cognitive-demand math tasks and maintain the rigor of mathematical tasks throughout lessons and units.

Emphasis #3-Developing beliefs: Instruction should support students in developing beliefs that mathematics is sensible, worthwhile, and doable.

Emphasis #4-Engaging students in mathematical practices: Instruction should provide opportunities for students to engage in multiple mathematical practices, including, but not limited to, the eights listed in the CCSS: 1) Make sense of problems and persevere in solving them, 2) reason abstractly and quantitatively, 3) construct viable arguments and critique the reasoning of others, 4) model with mathematics, 5) use appropriate tools strategically, 6) attend to precision, 7) look for and make use of structure, and 8) look for and express regularity in repeated reasoning.

We can see from these emphases that students should be focusing on making sense, using multiple representations of mathematical concepts, communicating their thought processes, and justifying their reasoning. Several of the CCSS mathematical practices involve language and discourse (in the sense of talking, listening, reading, or writing), in particular math practices #3 and #8. In order to engage students in these mathematical practices, instruction needs to include time and support for mathematical discussions and use a variety of participation structures (teacher-led, small group, pairs, student presentations, etc.) that support students in learning to participate in such discussions.

According to a review of the research (Hiebert & Grouws, 2007), mathematics teaching that makes a difference in student achievement and promotes conceptual understanding in mathematics has two central features: one is that teachers and students attend explicitly to concepts, and the other is that teachers give students the time to wrestle with important mathematics. Mathematics instruction for students learning the LOI should follow these general recommendations for high-quality mathematics instruction and focus on mathematical concepts and the connections among those concepts while using high-cognitive-demand mathematical tasks, for example, by encouraging students to explain their problem-solving and reasoning (AERA, 2006; Stein, Grover, and Henningsen, 1996).

One word of caution: concepts can often be interpreted to mean definitions. However, paying explicit attention to concepts does not mean that teachers should focus on providing definitions. Instead instruction can focus on students developing meaning for important mathematical concepts (e.g. equivalent fractions or the meaning of fraction multiplication). Similarly, students can show their conceptual understanding not by giving a definition or describing a procedure, but by using multiple representations. For example, students can show conceptual understanding of equivalent fractions by using a picture of a rectangle as an area model to *show* that two fractions are equivalent or a picture of fraction multiplication to show why multiplication by a positive fraction smaller than one makes the result smaller. Such pictures can be accompanied by oral or written explanations that students have opportunities to first construct on their own, then share with a peer, in small groups, and later in a while class discussion. Students will also need feedback and opportunities to revise these drawings and oral or written explanations.

The preceding examples point to several challenges that students face in mathematics classrooms focused on conceptual understanding. Since conceptual understanding is most often made visible by showing a solution, describing reasoning, or explaining the mathematical "why," instead of simply providing an answer, this shifts expectation for students from carrying out procedures to communicating their reasoning. Students are expected to a) communicate their reasoning through multiple representations (including objects, pictures, words, symbols, tables, graphs, etc.), b) engage in productive pictorial, symbolic, oral, and written group work with peers, c) engage in effective pictorial, symbolic, oral, and written interactions with teachers, d) explain and demonstrate their knowledge using emerging language, and e) extract meaning from written mathematical texts.

The main goals for teachers teaching mathematics to students who are learning the LOI are to teach for understanding, support students in using multiple representations, and support students as they use emerging and imperfect language to communicate about mathematical concepts. Since many resources describe how to teach mathematics for understanding and use multiple representations, the recommendations below focus on how to connect mathematical content to language, in particular through "engaging students in mathematical practices" (Emphasis #4).

RECOMMENDATIONS FOR CONNECTING MATHEMATICAL CONTENT TO LANGUAGE

Recommendation #1: Focus on students' mathematical reasoning, not accuracy in using the LOI

Instruction should focus on uncovering, hearing, and supporting students' mathematical reasoning, not on accuracy in using language (either the LOI or a student's home language). When the goal is to engage students in mathematical practices, student contributions are likely to first appear in imperfect language. Teachers should not be sidetracked by expressions of mathematical ideas or practices expressed in imperfect language. Instead, teachers should first focus on promoting and privileging meaning, no matter the type of language students may use. Eventually, after students have had ample time to engage in mathematical practices both orally and in writing, instruction can then carefully consider how to move students toward increasing linguistic accuracy in the LOI.

As a teacher, it can be difficult to understand the mathematical ideas in students' talk in the moment. However, it is possible to take time after a discussion to reflect on the mathematical content of student contributions and design subsequent lessons to address these mathematical concepts. But, it is only possible to uncover the mathematical ideas in what students say if students first have opportunities to participate in discussions and if these discussions are focused on mathematical ideas. Understanding and re-phrasing student contributions can be a challenge, perhaps especially when working with students who are learning the LOI. It may not be possible to decide whether what a student says is due to the student's conceptual understanding or to the student's language proficiency in the LOI. However, if the goal is to support student participation in a mathematical discussion and in mathematical practices, determining the origin of an error is not as important as listening to students and uncovering the mathematical ideas in what they are saying.

Recommendation #2: **Shift to a focus on mathematical discourse practices, move away from simplified views of language.**

In keeping with the focus on *mathematical practices* (Emphasis #4) in mathematics education research and standards, the focus of classroom activity should be on student participation in mathematical discourse practices such as explaining, conjecturing, and justifying. Instruction should move away from simplified views of language as words, phrases, vocabulary, or a list of definitions. In particular, teaching needs to move away from simplified views of language as vocabulary and leave behind an overemphasis on correct vocabulary and formal language, which limits the linguistic resources teachers and students can use in the classroom to learn mathematics with understanding. Instruction needs to move beyond interpretations of the mathematics register as merely a set of words and phrases particular to mathematics. The mathematics register includes styles of meaning, modes of argument, and mathematical practices and has several dimensions such as the concepts involved, how mathematical discourse positions students, and how mathematics texts are organized.

Another simplified view of language is the belief that precision lies primarily in individual word meaning. For example, we could imagine that *attending to precision* (CCSS mathematical practice #6) means using two different words for the set of symbols "x+3" and the set of symbols "x+3 =10." If we are being precise at the level of individual word meaning, the first is an "expression" while the second is an "equation." However, attending to precision is not necessarily about using the perfect word; a more significant mathematical practice is making claims *about precise situations*. For example, if a student makes the claim "Multiplication makes bigger," a teacher might follow it with the question "When does multiplication make the result bigger?" Through discussing and reefing the claim, students may arrive at a more precise claim such as "Multiplication makes the result bigger when you multiply by a positive number greater than 1." Notice that when contrasting these two claims, precision does not lie in the individual words nor are the words used in the more precise claim formal math words. Rather, in this case precision lies in the *mathematical practice* of specifying when the claim is true. In sum, instruction should move away from interpreting precision to mean using the precise word, and instead focus on how precision works in mathematical practices.

There are multiple interpretations of the term "precision" and it is important to consider what we mean by *precision* for all students learning mathematics, since all students are likely to need time and support for moving from expressing their reasoning and arguments in imperfect form. However, it is essential for teachers of students who are learning the LOI to consider when and how to focus on precision for those students. Although students' use of imperfect language is likely to interact with teachers' own multiple interpretations of precision, we should not confuse the two. In particular, we should remember that precise claims can be expressed in imperfect language and that attending to precision by focusing on a single word will interfere with students' expressing their emerging mathematical ideas.

Recommendation #3: Recognize the complexity of language in math classrooms and support students in engaging with that complexity

Language in mathematics classrooms is complex and involves a) multiple modes (oral, written, receptive, expressive, etc.), b) multiple representations (including objects, pictures, words, symbols, tables, graphs, etc.), c) different types of written texts (textbooks, word problems, student explanations, teacher explanations, etc.), d) different types of talk (exploratory and expository), and e) different audiences (presentations to the teacher, to peers, by the teacher, by peers, etc.). Language needs to expand beyond talk to consider the interaction of the three symbol systems involved in mathematical discourse--natural language, mathematical symbol systems, and visual displays. Instruction should recognize and strategically support students' opportunity to engage with this linguistic complexity.

Instruction needs to distinguish among multiple modalities (written and oral) as well as between receptive and productive skills. Other important distinctions are between listening and oral comprehension, comprehending and producing oral contributions, and comprehending and producing written text. There are also distinctions among different mathematical domains, genres of mathematical texts such as word problems, theorems, or textbook examples. Instruction should support movement between and among different types of oral and written communication including homework, diagrams, or textbooks, and different audiences (i.e. interactions between teacher and students or interactions among students). Instruction should recognize a) language as a complex meaning-making system and b) the complex nature of mathematical communication and activity as using multiple modes and sign systems (Gutierrez et al., 2010; O'Halloran, 1999, 2000; Schleppegrell, 2010).

Recommendation #4: Treat everyday language and experiences as resources, not as obstacles

Everyday language and experiences are not necessarily obstacles to developing academic ways of communicating in mathematics classrooms. It is not useful to dichotomize every day and academic language. Instead, instruction needs to support students in connecting the two ways of communicating by building on everyday communication and, when necessary, explicitly contrasting the two. Rather than debating whether an utterance or discussion is or is not mathematical discourse, teachers should instead explore what talk and visual displays mean to students and how they use these to make sense of mathematical ideas. Instruction needs to a) shift from monolithic views of mathematical discourse and dichotomized views of discourse practices and b) consider every day and mathematical discourses as interdependent, dialectical, and related rather than assume they are mutually exclusive.

Recommendation #5: Uncover the mathematics in what students say and do

Teachers need to learn how to recognize the emerging mathematical reasoning learners construct in, through, and with emerging language. In order to focus on the mathematical meanings that learners construct rather than the mistakes they make or the obstacles they face, curriculum materials and professional development need to support teachers in learning to recognize the emerging mathematical reasoning that learners are constructing in, through, and with emerging language (and as they learn to use multiple representations).

Teachers need support in developing the following competencies (Schleppegrell, 2010): using talk to effectively build on students' everyday language as well as developing their academic mathematical language; providing interaction, scaffolding, and other supports for learning academic mathematical language; making judgments about defining terms and allowing students to use informal language in mathematics classrooms, and deciding when imprecise or ambiguous language might be pedagogically preferable and when not. Materials and professional development should support teachers so that they are better prepared to deal with language and mathematical content, in particular a) how to uncover the mathematics in student contributions, b) when to move from every day to more formal ways of communicating mathematically.

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INVESTIGATING THE NATURE OF GRADE SIX AFTER SCHOOL MATHEMATICS CLUB LEARNERS' SHIFTS IN MATHEMATICAL NUMBER SENSE AND PROCEDURAL FLUENCY

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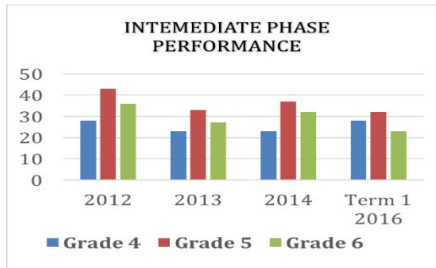
A wide range of research locally points to intermediate phase learners having extremely weak basic number sense resulting in the dominance of inefficient strategies for calculations, irrespective of the number range. In this paper I present research findings that suggest that learner workings when used in conjunction with visual progression spectra can provide important clues to researchers and teachers. This in turn contributes to an understanding of where learners are in their mathematical learning which gives ideas for how to support and progress them using more flexible methods. Included, is the discussion of the effectiveness of the club space to enable shifts in learners' number sense, learner flexibility, fluency and performance as displayed in learner methods and scores of the pre- and post- assessments.

CONTEXT AND BACKGROUND

South Africa is significantly underperforming in mathematics education at both primary and secondary levels. The condition of mathematics education in South Africa has often been described as being in crisis (Fleisch, 2008). Currently learners are unable to move their thinking sufficiently forward from concrete counting to abstract thinking (Graven and Stott, 2012). Poor quality of mathematics teaching, teacher knowledge, language, opportunities to learn, teaching time, home resources, and learner dispositions have been cited among other reasons as the cause of these challenges we are faced with in South Africa. Hence there is a need to strengthen in-service training of mathematics teachers and learner achievement in mathematics (Carnoy et al., 2008; Heyd-Mezuyanim and Graven, 2016; Hoadley, 2012; Reddy, 2006; Spaull, 2013).

The last two surveys conducted by SACMEQ, highlighted that South African learners performed below the poorer African countries such as Kenya, Tanzania and Swaziland (Spaull & Kotze, 2015). South Africa's overall performance in Rasch Scores in SACMEQ IV was 552 in mathematics and for the first time achieved above the mean SACMEQ score of 500. Despite South Africa making such improvement, less learners achieved higher mathematics competency levels and could cope with questions of higher cognitive demand (DBE, 2017).

Furthermore, the 2012 to 2014 Annual National Assessments (ANA) results reflected that learner performance in mathematics worsens in Grade 6 (DBE, 2014 p.92). In the same report it was revealed that in the Eastern Cape province, Grade 6 learners achieved below 40% (DBE, 2014). This situation is reflected in the Uitenhage District of Nelson Mandela Bay Metro where this study was carried out and the results for this district are shown in



the graph below.

Figure 1: Uitenhage district ANA 2013 – 2014 learner results & 2016 (Term 1) learner performance

The graph in Figure 1 above, shows not only the poor performance of the grade 6 learners that are being investigated in this study but also that of the grades four and five learners as well. The purpose of that is to show that from Grade 4 learners also obtain low marks. The situation seems to improve in grade five but worsens again in Grade 6. This 2012 – 2014 ANA results report mentions that 16% of Grade 3 learners have performed at a Grade 3 level in mathematics. Meaning that even at grade 3 level the vast majority are under performing in mathematics (DBE, 2014). Consequently, the problem of poor performance in mathematics seem to begin in the foundation phase and spills over to the intermediate phase.

The Mathematics Education Chairs Initiative (MECI) was established in 2010 in which six university academics were appointed to research Chairs in South Africa. The principal goals of the MECI initiative is to improve mathematics teaching and learning in schools, to broaden participation and improve performance in mathematics for improved economic competitiveness and wider social development. The rationale underpinning this initiative is for the Chairs to improve mathematics teaching and learning in the public schooling system through a close partnership with a selection of schools for the duration of their projects (MECI, 2018).

The SANC project, situated at Rhodes University in the Eastern Cape of South Africa, is one of two such numeracy chairs. Among other goals they focus on training and development for practicing (in-service) teachers to improve the quality of their teaching and the mathematics results in schools; to research sustainable and practical solutions to the mathematics challenges in the country; and to provide leadership on mathematics education and increase dialogue around the solutions (MECI, 2018).

One of the key objectives of the SANC project is to support the learners to make sense of numbers and to progress them from using inefficient and constrained to more fluent and flexible methods through after school Club activities. This push towards increasingly efficient methods of computing in mathematics and number sense is the central aspect of the project's Pushing for Progression (PfP) programme which is also a teacher development programme.

The PfP programme is a 15-week programme which aims to provide support for teachers to run weekly afterschool mathematics Clubs in their schools. The programme aims to develop learner sense making in numbers, shifting learner mathematical fluency from being passive learners to becoming active participants in their mathematics learning (South African Numeracy Chair Project, 2016). Teachers had three workshops sessions with the SANC project team members taking them through the resources and activities that are done during the afterschool mathematics Club sessions.

RESEARCH DESIGN

The Clubs formed the empirical field for this research. A social constructivist perspective of learning guided this study. Especially Vygotsky's (1978) notion that cognitive development stems from social interactions and guided learning within the Zone of Proximal Development (ZPD) of children, guided by more knowledgeable others. Furthermore, Kilpatrick et al.'s (2001) strands of mathematical proficiency provide the conceptual frame with a particular focus on procedural fluency and number sense.

A mixed method approach to data collection was used. Quantitative data has been drawn from learner's scores on pre- and post- assessments on four basic operations. Individual learner data was grouped together for a particular Club, enabling Club averages to be worked out. Similarly, all learner data was grouped together to enable overall PfP programme averages and changes to be calculated. Visual progression spectra have been adopted from the Pushing for Progression (PfP) programme which is an intervention programme developed by the SANC project for Club facilitators.

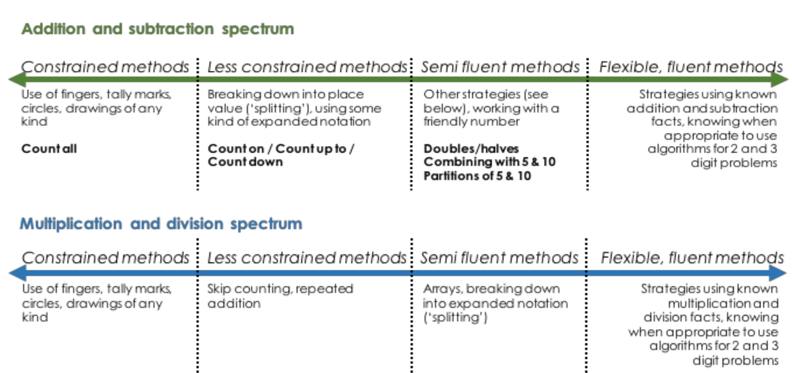


Figure 2: Addition/subtraction and multiplication/division spectra for the PfP programme

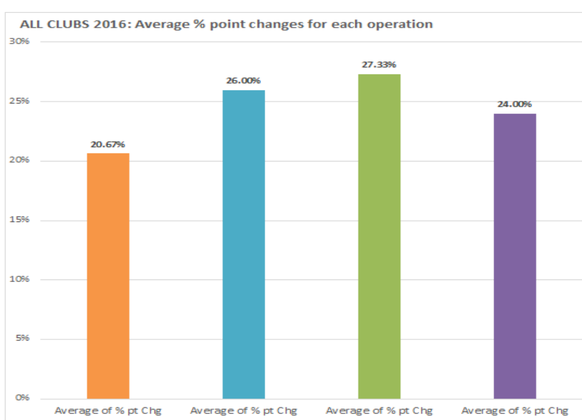
These spectra provide explanations of learner progression trajectories and how to analyse learner methods. Qualitative narratives were drawn from learner progression data, as well as teacher post Club questionnaires and one-to-one teacher interviews.

FINDINGS

Scores and percentage changes

The Club learners show the biggest average percentage point change in multiplication (27%), although there are pleasing changes across all operations, with an overall average change of 25 percentage points (Figure 4 below). This reveals that Club learners on average successfully completed five more questions (5 out of a possible total of 20 is 25%) in the post-assessment than in the pre-assessment. The graph in Figure 3 below is a summary of the scores for all 60 Club learners across the five Clubs.

57 out of 60 learners who participated in the Clubs show positive change between the pre- and post-assessment, ranging between -5 and 80 percent. This is reassuring given that at the start of the 15-week programme these learners attained scores that ranged between 0%



and 15% when the pre- assessment was administered.

Figure 3: Overall average percentage score changes for all 60 Club learners across the five Clubs

The number range of the items in the assessment is from 1 to 3 digits which is far less than the 6 to 9 digits that these learners are expected to compute proficiently by this grade. As these Club learners are in Grade 6, they should, according to the curriculum, be able to achieve 100 percent for this assessment.

Individual Club changes

Club A has an overall average change of 43%. This is relatively evenly spread across the 4 operations, although the biggest increase is with 55% in multiplication. Table 1 below shows the scores for each operation in each of the five Clubs as well as their overall average change.

Table 1: Summary of % point change in each Club by operation

Club	Average % point Chg. Addition	Average % point Chg. Subtraction	Average % point Chg. Multiplication	Average % point Chg. Division	Average Overall % point Chg.
Club A	32%	47%	55%	40%	43%
Club B	42%	38%	33%	42%	39%
Club C	0%	0%	22%	17%	10%
Club D	27%	10%	3%	-5%	9%
Club E	3%	35%	23%	27%	22%
Total	21%	26%	27%	24%	25%

Learner method findings

I conducted a micro analysis of methods used by the Club learners by comparing both pre- and post- scripts for all Club learners. The figures in Table 2 have been calculated by analyzing only the responses for *correct* answers from all learners in each of the 4 operations. There are 240 possible responses per Club per problem (12 Club learners x 20 items = 240 responses). Table 2 below presents the summary of Club learners' responses for each Club.

Table 2: Summary of the change in Club learners' methods between pre- and post-assessments

Number of correct responses…
of 240 possible responses

	with no shown workings			using non-tally methods			using tally methods		
	Pre	**Post**	**Change**	**Pre**	**Post**	**Change**	**Pre**	**Post**	**Change**
Club A	51	137	86	30	56	26	4	0	-4
Club B	48	27	-21	37	155	118	0	0	0
Club C	54	83	29	8	10	2	19	11	-8
Club D	59	65	6	25	56	31	9	8	-1
Club E	62	92	30	26	57	31	18	0	-18

Overall Averages	54.8	80.8	26	25.2	66.8	41.6	10	3.8	-6.2

The overall averages show an increase in no shown workings (+26) and the use of tally methods has gone down by 6.2 on average. The biggest change is in the use of non-tally methods with an average increase of +41.6 responses.

Detailed learner vignettes providing deeper analysis into the above data

To support the data showing learners' score changes over time, I share a richer, more detailed data from selected learners, derived from their workings on the assessment scripts. As Stott and Graven (2012) argue, progress in procedural fluency cannot be gauged only by looking at a single item from an assessment or by looking purely at scores. Indeed, fluency and number sense are defined in terms of ways of working with number and operations and not in terms of accuracy of answers. Therefore, I include a summary of the methods used by selected learners, across all 20 assessment items in both the pre and post-assessments (see Table 3 below). This allowed me to illuminate the overall shifts in methods used by the learners between the pre and post-assessments. I have selected several examples from learners written workings across the five Club sites to show the nature of learner progression in terms of fluency in methods. I chose those Club learners that progressed in scores and in more flexible methods used.

Table 3: Summary of number of responses for each spectra level for selected learners across all 20 assessment items (pre and post-assessment)

	←				
	Constrained[3]	**Less constrained**	**Semi fluent**	**Flexible, fluent**	**No. responses to assessment items**
Learner A1					
Pre	18			2	20
Post	2	3	4	11	20
Learner A6					
Pre	18	2			20
Post		2	6	12	20
Learner C27					
Pre	9	1		10	20

[3] Refer to Figure 2 for descriptors.

Pos t	7	2		11	20
Learner B13					
Pre	5	3	2	1	20
Pos t	5	5	5	4	20

The spectra incorporate elements of both number sense and procedural fluency, hence learners were placed at a particular level if the method used reveals lack of number sense and/or fluency and flexibility. For example, these responses in figure 4 below from **Learner A1 from Club A.**

Learner A1's pre-assessment, show a lack of number sense and they have been allocated as **constrained methods**.

104 + 97 = 20000	8 − 2 = 4
12 x 4 = 35	6 ÷ 3 = 4

Figure 4: Example - Learner A1 pre-assessment responses

In the post-assessment, the types of methods used are more diverse, with a significant increase in flexible, fluent methods from two to 11 responses. I share this learner's responses to the five subtraction items to illustrate this shift. Also, the time it took the learner to both assessments. Note that the learner spent 10 minutes less on the post-assessment than the pre. The shorter time indicates that perhaps the learner is more fluent and has a shifted number sense.

Table 4: Learner A1 scores on operations and time spent on assessments

	Assessment completion time	Addition	Subtraction	Multiplication	Division	Total score
Pre	27 minutes	2	0	1	0	3/20 (15%)
Post	17 minutes	3	5	4	5	17/20 (85%)

In the post-assessment, however the first three subtraction items were answered correctly, without any shown workings, perhaps indicating some recall of facts which were now known. The learner used an open number line representation to approach the two more difficult questions with a bigger number range i.e. 467 – 43 and 305 – 97 as in figure 5 below.

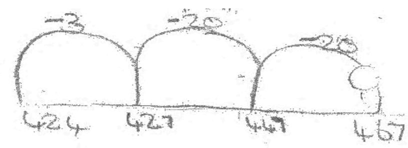
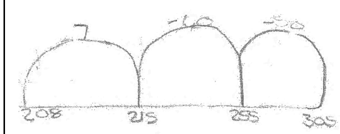
Item 2.4 post-assessment	Item 2.5 post-assessment
	

Figure 5: Learner A1 – Subtraction questions 4 & 5 (items 2.4 & 2.5)

For item 2.4, the learner broke the 43 into 20, 20 and 3 and jumped backwards on the number line to get to the correct answer of 424. For item 2.5, the learner broke 97 into 50, 40 and seven thus partitioning the number into chunks of tens then units. Item 2.4 could be calculated mentally, as no ‘borrowing’ is required, so the number line method is perhaps unnecessary. But the method does reveal an understanding of the structure of 43 as a number (20, 20 and 3). Similarly, item 2.5 could be calculated using rounding off (bridging to the nearest 100) and compensation strategy (305 – 100 = 205, add on 3 to get 208), but the number line method does show an understanding of the structure of 97 and its place value components. These methods reveal number sense in that both number lines show the correct direction of movement backwards for subtraction.

In looking at the spectrum for addition and subtraction in figure 2, I could say that this learner has moved from a constrained approach in the pre-assessment to somewhere between a semi-fluent and flexibly, fluent position in the post-assessment (with 15 responses overall). The learner knows some basic facts (showing fluency) and has another strategy (using a number line) for working out the harder items (exhibiting some number sense and flexibility).

Learner A6 from Club A

This learner progressed in all operations with a 60% percentage point change overall (see Table 5). She spent11 minutes less on the post-assessments.

Table 5: Learner A6 scores on operations and time spent on assessments

	Assessment completion time	Addition	Subtraction	Multiplication	Division	Total score
Pre	37 minutes	2	2	3	1	8/20 (40%)
Post	26 minutes	5	5	5	5	20/20 (100%)

While looking at both the pre- and post- assessments scripts for this learner I noted some progression in the methods she used (see figure 6). The pre-assessment script shows that this learner predominantly used constrained methods (18 responses) to answer questions. In the post-assessment, no constrained methods are visible and there is a large increase in flexible, fluent methods (12 responses). With this learner too, the shorter time perhaps indicates that the learner is becoming more fluent and has a strengthening number sense.

I used the multiplication items (figure 6) from Learner A6 to show shifts. In the pre-assessment, the learner used a vertical column method to answer all the items. The first three items were correctly answered. However, 24 x 6 and 120 x 15 were incorrectly answered. It is difficult to ascertain where the learner went wrong in using the vertical method. Although it could be argued otherwise, for these numbers it might be quicker to answer both of these questions mentally.

In the post-assessment, item 3.4 (24 x 6) was correctly answered using the vertical method and the learner successfully used a grid multiplication method to solve item 3.5 (120 x 15).

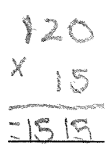
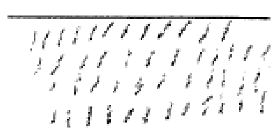
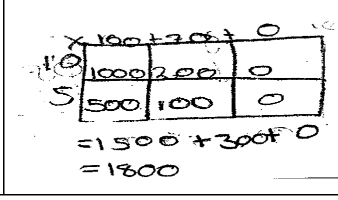
Item 3.5 Pre-assessment	Item 3.5 Post-assessment
 and 	

Figure 6: Learner A6 - Multiplication question 5 (item 3.5)

Although the grid method used for post-assessment item 3.5 could be considered a less fluent method as the answer could be worked out mentally[4], it displays improved accuracy, deeper number sense, particularly in place value with the larger number range and fluency in known facts of multiplying by 10. The variety of responses (known facts, vertical column and grid method) used by this learner for answering these multiplication items in the post-assessment reveals an improved flexibility in selecting different methods or strategies to solve the different items, rather than simply relying on the vertical column method for everything.

Looking at this learner's complete set of responses for multiplication I suggest that in the pre-assessment, the learner was fluent in some basic facts but working with a larger number range became tricky with these limited strategies. Thus, I could place the learner at the constrained end of the spectrum. In the post-assessment, I see a shift towards semi-fluent / flexible fluent methods overall.

Learner C27 from Club C

This learner progressed in all operations with a 30% percentage point change overall (see Table 6). S/he spent on 13 minutes less on the post-assessments.

[4] using 120 x 10 = 1200; half of 120 = 600, giving a total of 1800

Table 6: Learner C27 scores on operations and time spent on assessments

	Assessment completion time	Addition	Subtraction	Multiplication	Division	Total score
Pre	38 minutes	5	2	2	1	10/20 (50%)
Post	25 minutes	5	4	4	3	16/20 (80%)

While looking at both the pre- and post- assessments scripts I noted some slight progression in the methods s/he used (see figure 7). The pre-assessment methods show that this learner used an almost equal mix of constrained methods (9 responses) and flexible, fluent methods (10 responses). In the post-assessment, slight increases are seen in less constrained and flexible, fluent methods. Looking at multiplication and division, the first two items in each operation were correctly answered without shown workings, indicating that perhaps this learner knows their basic facts and can retrieve them.

In the pre-assessment, multiplication items 3.3 and 3.4 were calculated using a mixture of tally marks and a vertical method, which looks like addition rather than multiplication. The post-assessment methods of repeated addition in vertical columns, gives correct answers.

Notably the learner was still using a column method but reverted to using repeated addition. One could argue that this is a regression, especially for a Grade 6 learner. However, the repeated addition shows some level of developing number sense, whereas the pre-assessment method makes no sense at all.

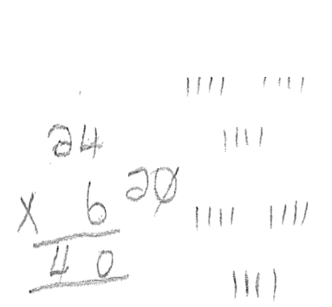
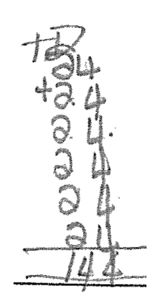
Item 3.4 Pre-assessment	Item 3.4 Post-assessment
	

Figure 7: Learner C27 - Multiplication question 4 (item 3.4)

Learner C27 incorrectly answered the fourth division question (item 4.4 in Figure 8) in the pre- assessment using a traditional algorithm. In the post- assessment, she drew 75 circles and counted these off in groups of three to correctly answer the question.

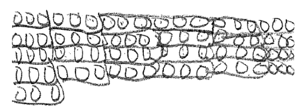
Item 4.4 Pre-assessment	Item 4.4 Post-assessment
75 ÷ 3 = 71	$75 \div 3 = 25$ 

Figure 8: Learner C27 - Division question 4 (item 4.4)

Even though the learner had a correct response in the post- assessment, we argue that this is not a fluent method for dividing 75 by 3. In Grade 6, learners are expected to use known division facts to answer this question. Alternatively, a learner with deeper number sense of division may use the inverse of known multiplication facts to get to the solution.

The responses to all the multiplication and division items in the pre-assessment reveal that the learner knows some basic facts which suggests fluency. However, the approaches used to answer the harder items indicates a lack of fluency. So, the learner could be said to be at a constrained level overall, for this grade. In the post-assessment, the knowledge of some facts remains (flexible fluency), interspersed with some semi-fluent methods for the harder items. One could place the learner at the semi-fluent position on the spectrum as both the multiplication and division methods used do show an emergence of sense making. The shorter time indicates that this learner too is more fluent and has a shifted number sense in computing some operations.

Learner B13 from Club B

Table 7 shows that this learner completed the post-assessment 8 minutes faster in the post-assessment and has a 50-percentage point change achieving full marks for addition, subtraction and multiplication in the post-assessment.

	Assessment completion time	Addition	Subtraction	Multiplication	Division	Total score
Pre	29 minutes	5	3	0	1	9/20 (45%)
Post	21 minutes	5	5	5	4	19/20 (95%)

Table 7: Learner B13 scores on operations and time spent on assessments

I noted from the learners' scripts that it was common to use the vertical procedure incorrectly and inappropriately in both the pre- and post- assessments especially as a method for the last three items of every operation. However, this learner approached multiplication item 3.5 in both assessments differently. In figure 9 below, he decomposed this number into place value horizontally in the post-assessment. The use of this method shows a deeper understanding that he must decompose the multiplicand and not the multiplier and he correctly uses the brackets.

Item 3.5 Pre-assessment	Item 3.5 Post-assessment
120 x 15 = (100 + 20) x (10 + 5) = (100 x 10) + (20 x 10) not plus = 2000 + 200 = 400 multip	120 x 15 = (100 x 15) + (20 x 15) = 1500 + 300 = 1800

Figure 9: Learner B13 – Multiplication question 5 (item 3.5)

Findings from teacher's experiences of learners changing mathematical proficiency and of working in the Club space

The data findings from the post club teacher questionnaire were themed into four themes namely learning environment, learning approach, learning participation and attitudes and finally changes in learners' procedural fluency and number sense. The teachers noted that the club learners are now keen to attempt work given to them. One can see from their workings the work was not copied from the way they have attempted the activity. Even in the classroom they answer without being probed, they also show efforts in doing both their class and homework. They respond and answer more quickly, counted and computed much faster. Eventually, club learners developed love for the subject.

They attributed these shifts to the relaxed atmosphere in the club space, small sized groups, learning through play with co members. They believed that such games provoked learner thinking and the entire environment helped develop the learners' number sense, fluency and flexibility. The clubs have made them ready for grade seven.

CONCLUSION

The overarching aim of this research was to investigate whether the Club could be a productive learning space for learners when Clubs are run by their teachers. I suggest that based on my findings these Clubs, run by teachers and not members of the SANC project team or outside volunteer facilitators, were a productive space for developing learners' procedural fluency and number sense. This is evident in the progress and shifts in procedural fluency and number sense shown by the data findings. I note with interest that 57 out of the 60 Club learners in the PfP programme of this study improved sufficiently to be promoted to grade seven at the end of 2016. Recall that the learners chosen for these Clubs were in the low performing pool. Teachers indicated that they bunked classes, had low morale and low or no confidence at all. It is compulsory in South Africa that learners pass mathematics with level three in order to progress to the next grade.

Baselining and profiling learner workings in the visual progression spectra has helped in realising where the learners where in their Mathematical learning and on how they can be progressed. While all five Clubs showed learner progress, it is worth nothing that some Clubs were more productive (in terms of improved scores and learners' methods) than others. This is to be expected as the Clubs were run by different Club facilitators with different mathematical knowledge and pedagogical approaches. Also, other contextual factors would differ slightly from Club to Club. The South African Intermediate Phase CAPS works on the assumption that teachers know all the methods and the activities to be done for mental maths prescribed in the curriculum. However, being part of the Club programme has raised the issue of this gap in teachers' knowledge and pedagogy. The programme has helped me as a District Official to realise the importance of my role in working towards strengthening teacher knowledge and closing such gaps.

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USING A CONSTRUCTIVIST THEORY TO FORMULATE GENETIC DECOMPOSITIONS FOR QUALITY MATHEMATICS TEACHING

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This plenary presentation is planned to suggest to colleagues from the Mathematics community an alternate approach to our teaching. We are reminded by the AMESA congress about the importance of Mathematics on the social, economic and technological advancements in our country and across the world. The constant dynamic changes which occur in the mentioned advancements of society, economy and technology demand that the Mathematics teachers adapt their curriculum accordingly. In the spirit of creating change in our teaching this presentation focuses on ideas already trial and tested and shown satisfactory results in enhancing students' Mathematics performance. I will present a way of questioning our teaching of Mathematics and display the theoretical aspects via three examples. These examples were chosen in accordance with the spread of the audience background. One example I will chose from the primary school syllabus, one from the high school syllabus and one from tertiary level of undergraduate Mathematics. The talk shall constitute three aspects: 1) I will present a brief on the curriculum research framework, 2) I will then present a theory that is used both as a curriculum designer and as an analytic tool and 3) I shall highlight these theoretical aspects with Mathematics examples from school and university.

INTRODUCTION

Mathematics student performance has been for decades recognised as a problem in society. This is the case in schools and higher education institutions, especially at the undergraduate level. It is thought that if one understands how students think when engaging in mathematics activities then one might be able to improve on ways of making the learning of mathematics more meaningful. This coincides with the congress theme: "**Developing Deep Mathematical Thinking through Mathematics Teaching**". Hence exploring student mathematical thinking is important not only to mathematics education research but the country and a global society as a whole.

I will present, what is called an ACE cycle, which entails an exploration on how an individual learns and the results of which inform our teaching and then implement these findings in later teaching. The exploration is a very empirical one and aligns itself with the congress sub-theme: "**Developing deep mathematical thinking through research in mathematics education**". The mathematics curriculum research framework I adopt is presented in Figure 1.

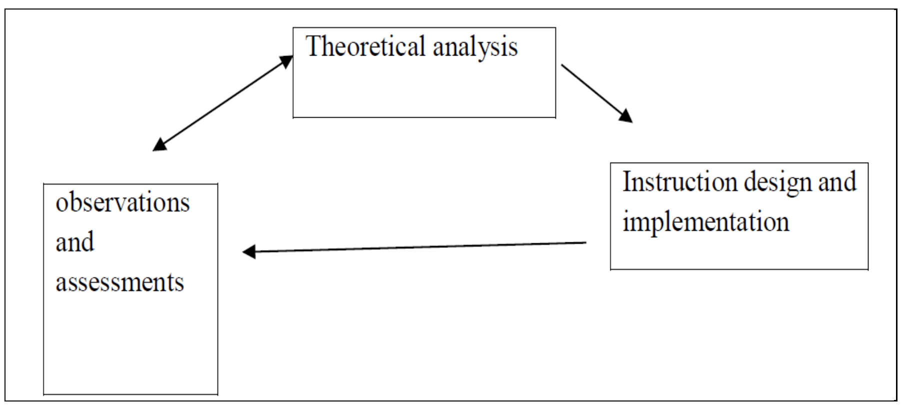

Figure 1: *The Research Frame work*

Source: Asiala et al., 1996

THE THEORETICAL ANALYSIS

APOS studies adopt a research framework postulated by Asiala, Brown, DeVries, Dubinsky, Mathews, & Thomas (1996). The theoretical analysis (see Figure 1) produces assertions about mental constructions that can be made in order to learn a specific mathematical topic, instruction tries to create situations which can foster making these constructions, observation and assessment tries to determine if the constructions appear have been made and the extent to which the student actually learned (Asiala et al., 1996). For this presentation I will consider theoretical analysis with a focus on mental constructions. Under theoretical analysis APOS theory of Dubinsky will be useful by which he describes the cognitive structures used by students to construct knowledge through action, process, object and schema. APOS theory is a theoretical framework for the process of learning Mathematics that pertains specifically to learning more advanced mathematical concepts (Weyer, 2010).

It is a theory that is premised on the hypothesis that mathematical knowledge consists of an individual's tendency to deal with perceived mathematical problem situations by constructing mental *actions*, *processes*, and *objects* and organizing them in *schemas* to make sense of the situations and solve the problems (Dubinsky & McDonald, 2008). According to Cooley, Loch, Martin, Meagher & Vidakovic (2006)' APOS allows for the development of ways of thinking about how abstract Mathematics can be assimilated and learned. Figure 1 shows the research framework for research and curriculum development. It is with this research framework that the theoretical framework adopted from Jojo, Brijlall & Maharaj (2013) fits theoretical analysis.

For the theoretical analysis stage of the research framework, the studies I refer to use APOS theory. For the Instruction design and implementation stage all these studies derive their formulation from the ACE cycle (Asiala et al. (1996). The A stands for activities, the C for classroom discussion and E for exercises done. This ACE teaching cycle is a pedagogical approach, based on APOS theory. It is a repeatable cycle and the activities designed are supposed to foster the students' development of the APO mental structures. This development depends on the opportunities for learning both in and outside the classroom. By engaging in these mathematical opportunities, students are guided by the teacher/lecturer to reflect on the activities and their relations to the mathematical concepts being taught. The teachers/lecturers then facilitate discussion in relation to the results produced by the students. This discussion involves the learner/student explaining herself/himself within the classroom on an individual basis or as from a group representative if collaborative learning has taken place. The classroom discussion is generally followed by homework exercises which would be fairly standard tasks similar to those encountered in class. These are intended to consolidate, reinforce and strengthen the work done in class and the discussion involving the students and teacher/lecturer.

The discussion occurring in the classroom and the analysis of written responses followed by interviews would comprise stage three titled observations and assessments in Figure 1. The findings emanating from this analysis could inform the APOS theory. Also note the two-directional implication between the theoretical analysis and observations and assessments stages in Figure 1. This would mean that APOS theory can be used as an analytic tool as well. This is generally done by firstly devising what is called a genetic decomposition (GD) which is discussed later.

In recent developments the focus on Mathematics learning is on the mental processes that an individual employs to understand a learnt concept (Brijlall & Ndlovu, 2013).

Itemized genetic decomposition

The genetic epistemology of Jean Piaget is found to be useful for this study. At the centre of Piaget's work is a fundamental cognitive process which he termed "equilibration" (Piaget, 1967). Through an application of the model of equilibration to a series of written tasks we are able to generate an account of the arrangements of component concepts and cognitive connections prerequisite to the acquisitions of the concept of the maxima and minima. These arrangements which are called "genetic decomposition" do not necessarily represent how trained mathematicians understand the concepts. Brijlall & Ndlovu (2013) introduced the notion of itemized genetic decomposition. An itemized genetic decomposition (IGD) they defined as a genetic decomposition specific to a mathematics task an individual is confronted with. For example they spoke of an IGD for rectangle area which dealt with a specific problem requiring learners to find the maxima/minima area of a given rectangle. We extend this notion of IGD in this study for two tasks which we highlight in the methodology section of this paper.

An example from the primary school syllabus

The CAPS (DoBE, 2011) spells out the scope for grade 7 in terms of geometric and algebraic patterns. The policy states that number patterns the aim is to investigate and extend patterns. In order to achieve this lessons have to be designed to: 1) explore and extend numeric or geometric patterns looking for relationships between numbers, 2) represent in physical or diagram form the patterns, 3) represent in tables learner's own creation of the number patterns and to 4) allow for learners to describe and justify the general rules for observed relationships between numbers in own their words. In this presentation an initial genetic decomposition is discussed for this aspect of teaching at grade seven.

An example from the primary school syllabus

For an example in high school I consider a task taken from Calculus which is tested in paper one of the senior certificate examinations. The problem is an optimization (see Frame 1) we extract from Brijlall & Ndlovu (2013).

A rectangular box has the following dimensions:

- Length 5x units
- Breath (9-2x) units
- Height x units

1. Write down the formula for finding the volume of the box.

2. Hence find the volume in terms of x.

3. Find the value of x for which the box will have a minimum value.

4. Explain all the steps you followed when finding the minimum value.

Frame 1: The problem statement for the task

Source: Brijlall & Ndlovu (2013)

We make use of the IGD constructed in that paper as an illustration on how one can design IGD for high school mathematics tasks.

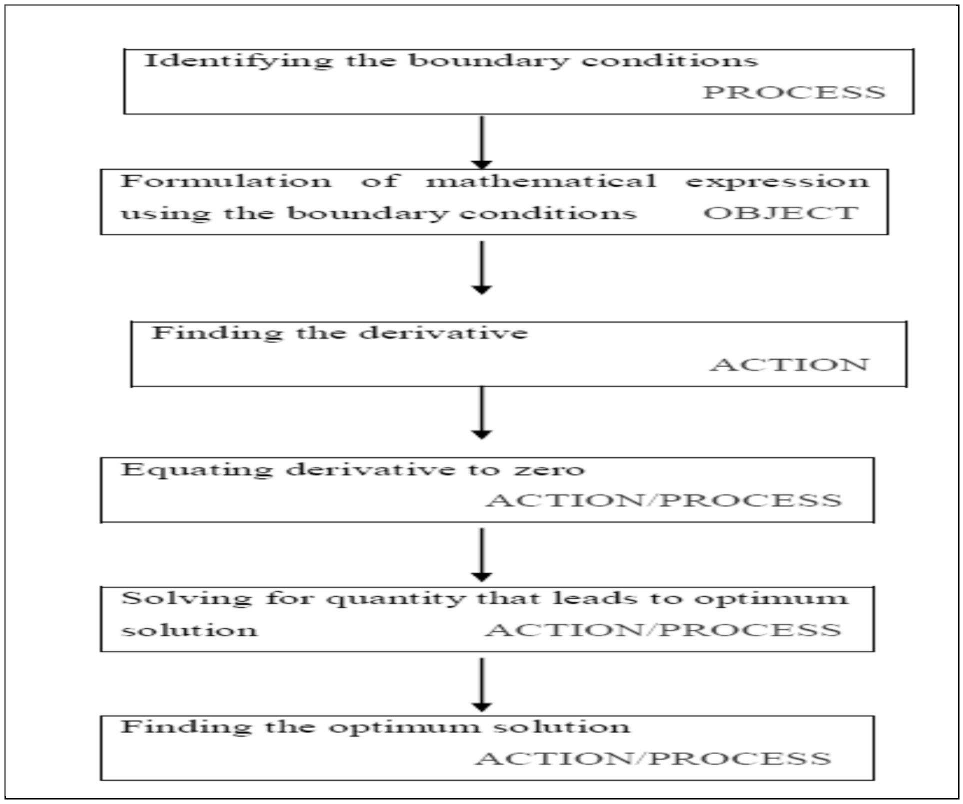

Figure 2: *An IGD for volume of cube*

Source: Brijlall & Ndlovu (2013)

An example at the tertiary level

For an example from undergraduate university Mathematics I present a modified genetic decomposition for techniques of integration in Figure 3.

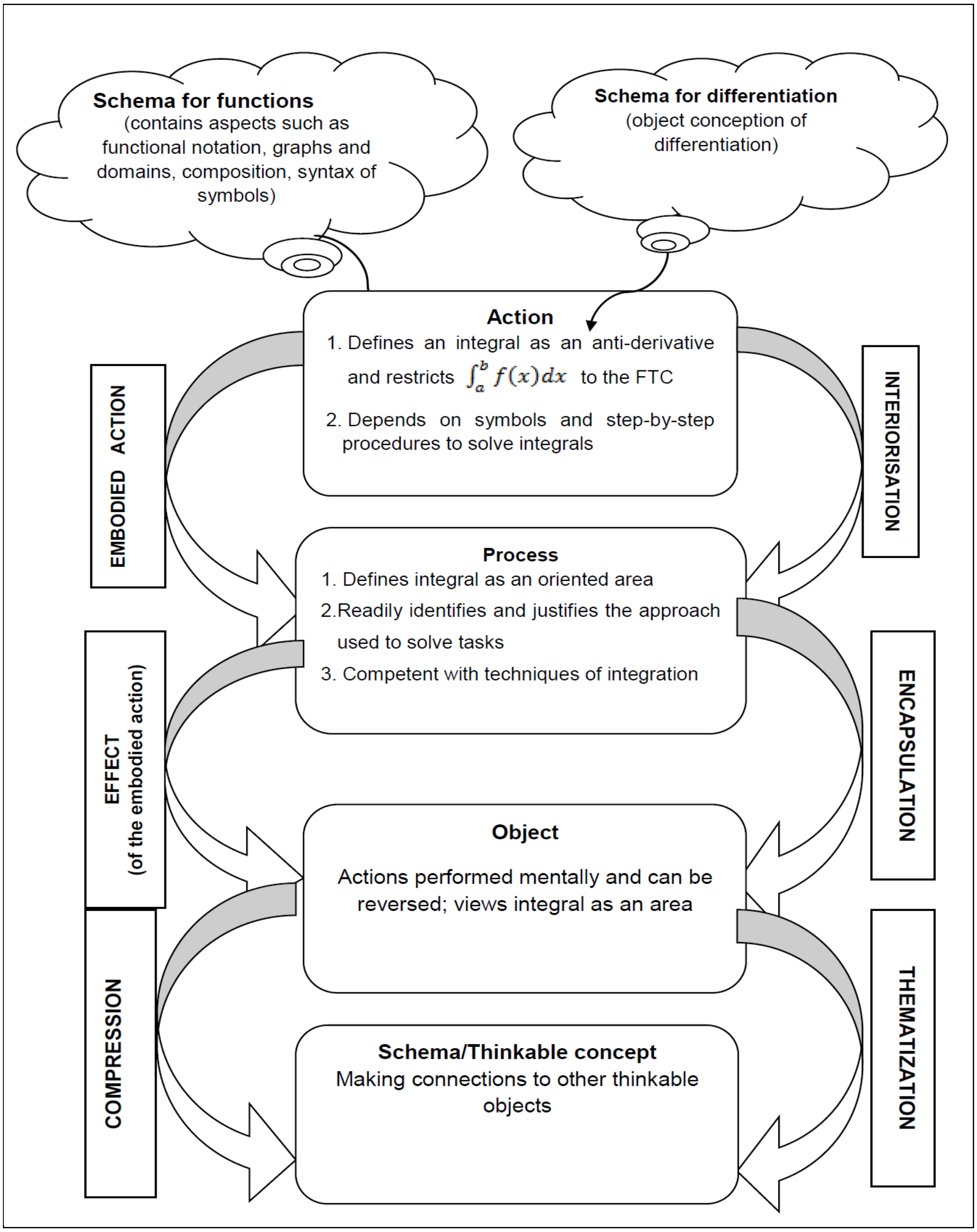

Figure 3: *A proposed genetic decomposition for integration*

Source: Brijlall & Ndlazi (2019)

CONCLUSION

This presentation has discussed an approach when reflecting on our teaching via exploring how learners/students understand their mathematics. We firstly arrange a genetic decomposition on a topic from our experience as teachers and text books and other relevant materials. Thereafter, we use empirical findings to better adjust the preliminary genetic decomposition and formulate a modified/refined genetic decomposition for future teaching. An example of a preliminary genetic decomposition was presented in the case for primary school teaching which of course has not been expired yet. I presented modified genetic decompositions for examples from high school and tertiary mathematics.

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PROFESSIONAL DEVELOPMENT IN MATHEMATICS AND PHYSICAL SCIENCES USING DIAGNOSTIC SELF-EVALUATION

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RP Wesi
Sasol Foundation
N Van der Westhuyzen
Siyavula

Self-evaluation is a critical aspect of continued development that allows professionals to reflect on their own mastery and act on the insights for their own development and growth. The presentation will report on the use of a self-evaluation tool that allows any educator to evaluate their own content expertise by completing Mathematics and Physical Sciences questions in the comfort and privacy of their own spaces, using their own device, at a time convenient for them. This project is the initiative of the Department of Basic Education in partnership with the Sasol Foundation and Siyavula

A STORY OF DEEP MATHEMATICAL THINKING DRIVEN BY THE IDEA OF SYMMETRY

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In this talk we will look at a line of development in mathematics that begins with exploration of symmetry in the study of polynomial equations and the role it had in the development of abstract algebra, then moves on to the exploration of the concept of a bijection between sets seen as a symmetry, and its role in the development of abstract mathematics; the next encounter is the theoretical symmetry in the study of mathematical structures such as groups (algebraic structures that arise in the study of polynomial equations), and its role in the development of category theory. To connect the story with my own research in mathematics, the final encounter is my recent work (with collaborators) on the theory of forms, that proposes a symmetric approach in the study of the category of groups. No knowledge of these subjects is assumed. The main goal of the talk is to contemplate on the essence of "deep mathematical thinking".

EQUATIONS

It is well known that the quadratic polynomial equation $ax^2 + bx + c = 0$ has its solutions given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula goes back to the work of famous Persian scholar, Muhammad ibn Musa al-Khwarizmi, in the 9th century. Gerolamo Cardano, Italian scholar from 16th century, produced a formula (which is much longer and more complicated than the one above) for the cubic polynomial equation. René Descartes, French scholar from 17th century found (even more complicated) formula for the quartic polynomial equation. Common features of all of these formulas is that they are expressed using standard arithmetic operations including the roots. Norwegian mathematician Niels Henrik Abel from 19th century, who lived a short life of 26 years, demonstrated that such a formula does not exist for quintic polynomial equations. For example, there is no formula specifying the roots of the equation $x^5 - 4x + 2 = 0$. Does this mean the subject has been exhausted? Not at all! The next step is to try to understand why does such formulas not exist for higher order polynomial equations? The is precisely the question which was attempted by a young French mathematician, Éveriste Galois, towards the end of his short life of 21 years. He discovered that the existence of a formula for any particular polynomial equation is equivalent to a property of a mathematical structure called a "group" associated to the polynomial. Namely, this property is the possibility of decomposing the group into a special series of "subgroups". A group consists of "symmetries" – operations of matching a shape onto itself. In the case of polynomial equations, these operations match solutions of the equation onto itself. It is interesting to see how formal mathematical investigation of symmetry leads to understanding and clarification of the problem of finding formulas for solving polynomial equations. This subject is today known as Galois Theory and is usually taught in the honors level at universities in South Africa. Because of Galois Theory and subsequent developments, the study of groups is considered as one of the central areas of modern pure mathematics.

BIJECTIONS

Any group is a "subgroup" of a bigger group consisting of all "bijections" from one set to itself. The notion of a bijection between two sets captures the idea of symmetry at the most abstract level of thinking. A bijection is simply any matching of objects in one set to objects in another set that pairs each object from one set with exactly one from the other, in both directions. We may think of this as a mathematical transformation (function) that does not "distort" the information (which is just another way of thinking of a symmetry). When working with abstract sets, the information that is not distorted is the quantity of objects in the sets. Indeed two sets will have the same amount of objects if and only if there is a bijection between them. When dealing with finite sets this may seem as a rather obvious remark. However, the power of this remark is that it allows us to define what it means for two infinite sets to have the same amount of objects. Thus bijections allow us to compare infinite quantities. This was a realization of German mathematician from the second half of 19th century, Georg Cantor. He is the author of the quote: "In mathematics the art of proposing a question must be held of higher value than solving it." Cantor collaborated on "set theory" together with his colleague and friend, Richard Dedekind, who was interested in finding universal formalization of mathematical ideas. It turned out that the language of abstract sets was the right language for this. Yet another German mathematician, Ernst Zermolo, wrote down the first set of basic principles (axioms) for this new theory. The theory then flourished further and became indeed the first universal language for formalizing all mathematical ideas, including that of what is a "mathematical structure" of which a group is just one example.

DUALITY

With the development of set theory and mathematical logic, it becomes possible to define (mathematically) what a "mathematical theory" is. We can say that a theory is "symmetric", when it has a way of matching it onto itself without deforming the assumptions (axioms) of the theory. Unlike in Galois Theory, where a matching interchanges solutions of a given equation, here we interchange certain terms of the theory. This process is called "duality". For instance, in geometry, we may interchange the terms "point" and "line", and accordingly, the phrases "point lies on a line" and "line passes through a point". This would, however, distort the principles of ordinary plane geometry. Indeed, the statement that "for any two non-identical points there is a unique line passing through both", which is considered as one of the axioms, would now become "for any two non-identical lines there is a unique point lying on both" which contradicts the axiom stating existence of parallel lines. However, in what is called "projective plane geometry", this transformed statement is still an axiom. Thus, projective plane geometry exhibits duality. A well-known American mathematician of the 20th century, Saunders Mac Lane, remarked in his investigation in late 1940's, that "group theory" could perhaps also exhibit a duality similar to projective plane geometry. Group theory can be seen as a mathematical study of symmetry: the question now is whether such study itself is symmetric! In the investigation of this question Mac Lane came up with some of the most fundamental ideas in a subject called Category Theory that studies mathematical structures via their transformations, which he co-founded together with a Polish mathematician Samuel Eilenberg in 1940's. However, he only attempted the question in the case of the theory of so-called "abelian groups".

FORMS

For all groups, and many other group-like structures studied in modern mathematics, a decisive answer to the above question was obtained with the help of the "theory of forms" developed by the present author and his collaborators. An exposition of duality for group theory is given in his paper co-authored with a South African mathematician, Amartya Goswami and published in Advances in Mathematics in 2019. A "form" is an abstract system that describes an organization of mathematical structures, their transformations and their substructures. All groups, their transformations (called "group homomorphisms") and subgroups are organized in a certain form that exhibits symmetry. At the same time, an individual group also can be seen as a particular (very primitive) instance of a form (also exhibiting certain symmetry).

CONCLUSION

Behind each of the developments summarized above is the attempt to understand a given mathematical phenomenon, rather than the desire to solve a mathematical problem. This is the character of true "deep mathematical thinking". The process of solving a mathematical problem is an application of existing knowledge, whereas the process of understanding a mathematical phenomenon results in accumulation of new knowledge (which very often does result in development of new tools for solving problems – but then again, this is not its goal). In fact, when one engages in solving a new mathematical problem, the most effective approach is to try to understand the mathematical phenomena around the problem. Thus, a question arises: is it possible to develop deep mathematical thinking by learning and applying well developed algorithms for solving predetermined types of mathematical problems?.

EXPLORING THE ROLE OF THE MATHEMATICS EDUCATOR WITHIN THE FOURTH INDUSTRIAL REVOLUTION

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"The illiterate of the 21st century will not be those who cannot read and write, but those who cannot learn, unlearn, and relearn" Toffler (1970).

In recent years novel technologies have led to substantial transformations to our daily lives. It is probable that new technologies will continue to transform how we live for years to come. We have now infiltrated a new stage in the history of technological growth. Recent advances are starting to distort the lines concerning fact and fiction, providing indications that our connection with technology could be set to transform more radically than most of us may have predicted. This transformation is being labelled as the Fourth Industrial Revolution. In South Africa the Fourth Industrial Revolution is envisaged to create new job opportunities and a better society. However, amidst these innovative transformations, as educators of mathematics we need to discern the role that we will play within the Fourth Industrial Revolution. Thus, this presentation seeks to explore the role of the mathematics educator within the Fourth Industrial Revolution. Findings and discussions based on a national study will be presented.

Keywords: Educator; Fourth Industrial Revolution; Mathematics; Technology

WHAT DOES THE FOURTH INDUSTRIAL REVOLUTION MEAN?

The First Industrial Revolution occurred when we moved our dependence from animals, human effort and biomass as primary sources of energy to the use of fossil fuels and the mechanical power this enabled. The Second Industrial Revolution conveyed foremost developments in the form of electricity dissemination, both wireless and wired communication and new processes of power generation. The Third Industrial Revolution began in the 1950s with the development of digital systems, communication and rapid advances in computing power, which have enabled novel ways of creating and conveying information (Penprase, 2018).

The Fourth Industrial Revolution involves innovative capabilities for people and machines. While these capabilities are reliant on the technologies and infrastructure of the Third Industrial Revolution, the Fourth Industrial Revolution (4IR) represents novel methods in which technology becomes rooted within societies and human bodies (Schwab, 2016). Within the 4IR, robotics, the Internet of Things (IoT), virtual reality (VR) and artificial intelligence (AI) are transforming the way we live and work. Thus, as we embark on the Fourth Industrial Revolution, it is evident that technology will play a central role in nearly all aspects of our lives. However, the rapid pace of this transformation is disrupting industry and education in every country.

HOW DO WE PREPARE FOR THE FOURTH INDUSTRIAL REVOLUTION?

We ought to develop leaders with the skills to manage organizations through these extreme shifts. As professionals, we need to embrace this transformation. We need to understand that what our jobs are today might be diverse in the future. Our education and training systems ought to be modified to better prepare people for the flexibility and critical thinking skills they will need in the future workplace (Butler-Adam, 2018). Through the Department of Telecommunications and Postal Services, government is forging partnerships with key industry players in the Information, Communications and Technology (ICT) fields to implement programmes that prepare youth for the Fourth Industrial Revolution.

These programmes include the training of young people in various disciplines related to the Fourth Industrial Revolution such as coding and data analytics. Thus, the Fourth Industrial Revolution presents a set of challenges that educators need to address in order to convey pertinent instruction to contemporary learners.

The educator is tasked with the responsibility of guiding their learners towards the path of further study or the work environment. Both paths need to be carefully aligned with each learner's skills and capabilities. Teaching methods are evolving just as rapidly as the industries they serve, so we need to equip our educators with the latest innovative teaching methods. These methods will assist our educators to prepare our learners. To prepare the capacity needed for the digital economy, education must adapt as fast as the demand for ICT skills is growing and evolving. We need to ensure that educators complete programmes devised to help them acquire the set criteria of competency to enable them to become successful educators within the Fourth Industrial Revolution (Shan, 2018).

TEACHING MATHEMATICS IN THE FOURTH INDUSTRIAL REVOLUTION

Within the Fourth Industrial Revolution (4IR), it is crucial to enhance the learner's ability to problem-solve. Based on critical and creative thinking, problem-solving is important for flexibility and this is essential for survival within the 4IR. When teaching mathematics, educators need to ensure that learners are motivated to solve problems and in this way learners are expected to think about real world situations. To take advantage of 4IR opportunities, we need to transform our education, we need to adopt an interdisciplinary approach to Science, Technology, Engineering, and Mathematics (STEM) subjects. Mathematics, in particular, has been identified as a way of opposing unemployment and stimulating continued socio-economic existence. The research project under focus provides examples of how teacher educators may provide in-service and pre-service educators with opportunities to solve problems located within real world contexts. These problem solving techniques focus on employing the use of technology-based teaching methodologies.

THE FOURTH INDUSTRIAL REVOLUTION MATHEMATICS EDUCATION RESEARCH PROJECT

Being involved within Higher Education in the Fourth Industrial Revolution is a multifaceted and exhilarating prospect which can possibly improve society (Xing & Marwala, 2017). The project under focus was funded by the National Research Foundation (NRF) and employs the use of survey questionnaires, semi-structured individual interviews, lecture observations and semi-structured focus group interviews. The project was conducted at two universities in two provinces. More details regarding the methodology, analysis of findings and discussions will be explored during the presentation.

CONCLUSION

Nationally our universities are moving in a positive direction towards embracing the Fourth Industrial Revolution. Our pre-service and in-service educators are gradually being equipped with the skills necessary to teach within the Fourth Industrial Revolution. There is still much that can be done and teacher educators are becoming familiar with technology-based teaching methodologies for the Fourth Industrial Revolution. The project under focus will encompass data generated at an international university in 2020.

ACKNOWLEDGEMENT

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A MODELING APPROACH TO TEACH THE COMPOUND ANGLE FORMULAE FOR SINE AND COSINE

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In this paper, I look at the derivations of the compound angle formulae for sine and cosine provided in some of the textbooks prescribed for the National Senior Certificate which most of my first year students found to be unnecessarily complex to understand. In light of this, I propose, some alternative models of deriving these formulae with the learners' knowledge and level of understanding in mind. Each derivation uses only one basic concept for deriving each of the compound angle formulae. The beauty in using these models is that these derivations can be taught at Grade 11 and not necessarily at Grade 12 as prescribed in the Curriculum and Assessment Policy Statement (CAPS).

INTRODUCTION

I have observed significant gaps between knowledge and understanding of secondary school and undergraduate mathematics. Many of my students with good passes in the National Senior Certificate Examination (NSCE) experienced unexpected difficulties in their first-year mathematics at the university. This was revealed by studies and observations carried out with my first year students doing mathematics. Unless intervention methods are being put in place, students will continue to perform badly in the subject and drop out of mathematics at the end of their first year and migrate to other majors. One of the many reasons given is that many of the students have been inadequately prepared at school for meeting the demands of higher mathematics at tertiary level. I would agree with Mhlolo et al. (2012) that "there has been a general public discontent about learners' actual gains in knowledge and skills despite the steady increase in pass rates since the advent of democracy".

BACKGROUND OF MY STUDY

I have discovered several topics characterized by huge deficiencies amongst first year students due to superficial learning: Theory of Logarithms, Linear Programming, Calculus, Curve Sketching, Trigonometry, Probability & Statistics, and Financial mathematics. Unless these topics are re-taught as they are core to any undergraduate mathematics programme, many students will continue to struggle or filter out of any mathematics programme.

In this paper, I investigated the derivations of the compound angle formulae provided in some of the prescribed Grade 12 textbooks. Despite the NCS Grade 10-12 (Department of Education, 2003) clearly indicates that learners at Grade 12 are expected to derive and apply the compound angle identities for $\sin(\alpha \pm \beta)$,

$\cos(\alpha \pm \beta)$, sin2α and $\cos 2\alpha$, I have observed that many students doing first-year mathematics cannot derive these formulae, thus limiting them in the use of these formulae. Could it be for this reason that the proposed Curriculum and Assessment Policy Statement (CAPS) (Department of Education, 2011) has decided to dilute this part of the curriculum? In CAPS, it states that apart from accepting $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, learners are expected to derive the other compound formulae.

Through my interaction with a number of pre-service and in-service mathematics teachers, I found out that not much emphasis is placed on deriving the compound angle formulae, the reason being that many students find the derivations difficult and that these formulae are provided in the NSCE information sheet. They confessed that the compound angle formulae are taught as a set of rules and are then used in the derivations of other trigonometric identities. Only a few teachers (20%) derived the compound formula for $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. They followed the method as presented in their respective prescribed textbooks and, thereafter used it to derive the other remaining compound angle formulae. All my first-year students whom I interviewed so far have admitted that although these derivations are presented in the various textbooks, they found them difficult and preferred to learn them as a set of rules.

This article presents alternative and simple methods for deriving these formulae using a modelling approach where a basic knowledge/concept, which the students are already familiar with, has to be first identified for each derivation. Using this theory, each of the formulae is derived independently of one another using the identified concept as the base knowledge which is then expanded so that students can construct higher order knowledge and see the argument in each of the derivations. During the expansion phase the teacher acts as a guide or facilitator.

REVIEW OF TEXTBOOKS AND RELATED WORK

Several mathematics textbooks are prescribed for Grade 12 learners. In addition, almost all teachers use various "cook" books/study guides which contain solved examples only. However, only five prescribed textbooks are commonly used, the most popular one being *Classroom Mathematics* by Laridon et al. (2007). The derivations of the compound angle identities presented in each of these books are examined and discussed separately below.

Classroom Mathematics: The derivations of the compound angle formulae are discussed in Chapter 6, pages 137−138. The authors started with the derivation of $\cos(\alpha - \beta)$, which I shall decompose into three main steps that can be further broken down into mini steps (which makes it complex).

Step 1: Construction of a unit circle with centre (0, 0).

Step 2: Construction of some angles. Express the coordinates of the points on the circle in trigonometric form. Conclude congruency of two triangles. Deduce equality of each side of the triangles.

Step 3: Apply distance formula between two points to find the lengths of each of the two equal sides separately. Then equate the two lengths to derive the identity.

To derive $\cos(\alpha+\beta)$, they have used the fact that $\cos(\alpha+\beta) = \cos(\alpha-(-\beta))$. To derive the formula for $\sin(\alpha+\beta)$, they hinted that $\cos(90°-(\alpha+\beta)) = \sin(\alpha+\beta)$. To derive $\sin(\alpha-\beta)$, they expressed it as $\sin(\alpha+(-\beta))$. Though all these latter steps are correct, they do not actually encourage deep learning or in any way develop the strand of conceptual understanding of mathematical proficiency as discussed by Kalpatrick (2001). Deep learning occurs when conceptualization takes place in the learners' minds.

Study & Master Mathematics Grade 12 (Daan Van der lith, 2007) also discusses the compound angle formulae. The author uses a similar approach (page 225) to Laridon et al., but has added a rotation step, which increases the complexity of the derivation.

To derive $\cos(\alpha+\beta)$, the author equated $\cos(\alpha+\beta) = \cos(\alpha-(-\beta))$. To derive the formula for $\sin(\alpha+\beta)$, the author uses the fact that $\cos(90°-(\alpha+\beta)) = \sin(\alpha+\beta)$ and expressed $\cos(90°-(\alpha+\beta))$ as $\cos((90°-\alpha)-\beta)$ and equated it to $\sin(\alpha+\beta)$ after expansion. To derive $\sin(\alpha-\beta)$, he expressed it as $\sin(\alpha+(-\beta))$.

It is surprising to note that the author emphasized that learners are not required to derive these formulae but that they need to be able to use the results, which is contrary to the requirements of the NCS.

Another textbook, *Viva Mathematic, Learner's Book Grade 12* by Theron et al. (2007), provides a derivation for $\cos(\alpha-\beta)$ on page 199 using a unit circle and uses the Cosine rule to reach the conclusion as Laridon et al., but provide a much more simpler diagram. The other compound formulae are derived in the same way as the other authors.

Pretorius et al. in *Mathematics Plus, Grade 12* (Module 5, page 54) follow the same approach as that of Theron et al.

Finally, *Focus on Mathematics Grade 12* by Carter et al. (2007) present the derivation of $\cos(\alpha-\beta)$ as a learning activity (page 219) after providing a simple diagram in the first quadrant for learners to comprehend, and guiding them step by step to derive the formula using the Cosine rule.

All these authors have used a unit circle with centre (0, 0) but did not give any reason for doing so. This is one assumption that students and some teachers have failed to address.

My observations in the derivations of the compound angle formulae presented in these five books are that all the authors started with the formula $\cos(\alpha-\beta)$ and, thereafter derive the other formulae using procedural knowledge as followed by Laridon et al. In trigonometry, as in mathematics generally, learners at this level are usually taught in a "spiral" fashion: *sine*, *cosine* and then *tangent* so that learning takes place in a linear sequence. Similarly, the *sine rule* is taught before the *cosine rule*. So, why did these authors not start by deriving the compound angle formula for $\sin(\alpha + \beta)$? This is another motivation of my writing this paper. I believe that teachers should provide learners with the best learning trajectory. Trigonometry is introduced at Grade 10. The compound angles identities are taught in Grade 12, and are used across the mathematics curriculum at the undergraduate level in calculus, coordinate geometry and mechanics. So these formulae should be properly derived at the time they are introduced. Learners are not supposed to rote learn these formulae as a set of rules nor should they be taught to assume the compound angle for $\cos(\alpha - \beta)$ and then use it to derive other formulae.

The literature review would be incomplete if I do not refer to some textbooks used all over the world for the Pre-U Certificate in Mathematics/General Certificate of Education Advanced Level offered by the University of Cambridge International Examinations. Two such textbooks are: *Mathematics: The Core Course for A-level* by Bostock and Chandler (1992), and *Pure Mathematics 1* by Backhouse and Houldsworth revised by Horil (1988). Bostock and Chandler start by deriving the formula for $\sin(\alpha + \beta)$ on page 206. The method used is totally different from the South African textbooks and is relatively easier though the diagram used to derive this identity is quite complex. To derive the identity for $\cos(\alpha + \beta)$, learners are to replace α by $\left(\frac{\pi}{2} - \alpha\right)$ and, to derive the formulae for $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$, learners are to replace $\beta\ by\ (-\beta)$ in the addition formulae obtained.

The derivations for the compound angles identities are presented by Backhouse and Houldsworth in Chapter 17, page 341. They constructed two separate diagrams, one to derive the formulae for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$, and the other for $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$. However, both diagrams are quite complicated for learners to understand. I referred to these two textbooks because I have used them, teaching mathematics at Advanced level offered by Cambridge for more than a decade. I found them to be appropriate for undergraduate mathematics preparation and it was during my school teaching days that I decided to look for alternative ways to derive the compound angle identities.

A recent derivation for the compound angle formulae for *sine* from Ptolemy is presented by Michael de Villiers (2012). He concedes that his "derivations are no more difficult than the usual ones presented in South African mathematics textbooks, but have the added advantage of an interesting historical origin".

A geometric proof of the compound angle formula for sine is presented by Proobhalan Pillay (2017) using the trigonometric area rule. However, though his proof is interesting, it appears that the proof is restricted to acute angle triangles only.

THEORETICAL FRAMEWORK

In order to adequately prepare learners to meet the challenges of higher mathematics at tertiary level, high school mathematics teachers are faced by the daunting task of how to present information in class so as to develop the learners' critical thinking. Concerns have been raised that standards have actually dropped despite the upward trend in pass rates. Questions have also been asked to what extent mathematics high school teachers are creating opportunities for learners to develop their cognitive skill in problem solving (Mhlolo et al., 2012). Learning takes place in mathematics when learners are able to connect different ideas/concepts together to form a higher order concept or expand their knowledge. This is made easier when collaborative learning is encouraged in class. Learners from different backgrounds come with their own ideas and make different mathematical connections/contributions. In such a situation teachers play the role of facilitators who guide learners in their discussions and assist learners to recognize and make sense of their mathematical connections.

On the other hand, if learners are taught rules of thumb to learn mathematics, very little learning takes place. This is what Skemp (1978) refers to as "instrumental understanding". Learners know "what" to do without necessarily any knowledge of "why". More emphasis is placed on rules rather than understanding why such rules work. This is what Kilpatrick (2001) describes as "procedural knowledge". For learning with understanding to take place, learners are to know "what to do" and "why". This is relational understanding (Skemp, 1978) and high school teachers should prepare their learners to operate at this level, which Biggs and Collis (1982) refer to as the relational level in the SOLO Taxonomy. At the relational level learners are capable to make connections with bits of information presented to them and are able to appreciate the significance of the parts in relation to the whole.

It is, however, not uncommon in mathematics to expose learners to a number of methods/paths to get to the final solution/destination. But choosing a method/route which takes both minimum time and effort for learners to understand to get to the solution, to establish a proof or to derive a formula is very critical in the teaching and learning of mathematics at the high school level. Such a method/path does not encourage any superficial learning. On the contrary it motivates learners to explore or discover other methods to reach the solution/destination. However, the identification of the source concept is very crucial and is done by the teacher with the students involved. My proposed methods of deriving the compound angle formulae is to develop the relational understanding of learners by using some concepts which students are already familiar with and applied to some appropriate models. The teacher is to act as a facilitator guiding the learners to expand their existing knowledge.

In developing the appropriate models, I have recognized the findings made by Naidoo (2012) where “each master teacher incorporated the use of visual tools in order to make mathematical concepts easier to understand for the learners” and “these findings are important for advancing teacher and curriculum developments”. The main goal of teaching is the transfer of learning and as educators we teach learners for the future when we would not be there. If we are failing to add value to the information extracted from textbooks before presenting such information to the learners, then we cannot expect learners to develop critical thinking.

DERIVING THE COMPOUND ANGLE FORMULAE FOR SINE AND COSINE

Model 1

Construct Figure 1 to derive the compound angle formulae for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$

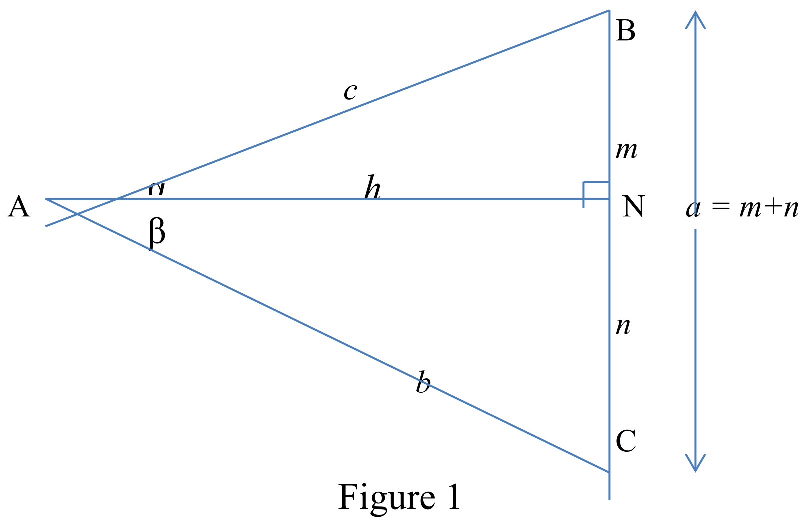

Figure 1

Solution:

$$Area\ of\ \Delta ABC\ =\ Area\ of\ \Delta ABN\ +\ Area\ of\ \Delta ACN$$

$$\Rightarrow \frac{1}{2}bc\sin(\alpha + \beta) = \frac{1}{2}ch\sin\alpha + \frac{1}{2}bh\sin\beta$$

$$\Rightarrow \sin(\alpha + \beta) = \left(\frac{h}{b}\right)\sin\alpha + \left(\frac{h}{c}\right)\sin\beta$$

$$\Rightarrow \sin(\alpha + \beta) = \sin\alpha.\left(\frac{h}{b}\right) + \left(\frac{h}{c}\right).\sin\beta$$

Hence $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

We shall use the same Figure 1 to show that

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Solution:

Let BC = a, BN = m and NC = n so that $a = m + n$

Using the Cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha+\beta)$$

$$\Longrightarrow (m+n)^2 = b^2 + c^2 - 2bc\cos(\alpha+\beta)$$

$$\begin{aligned}\Longrightarrow 2bc\cos(\alpha+\beta) &= b^2 + c^2 - (m+n)^2 \\ &= b^2 + c^2 - (m^2 + 2mn + n^2) \\ &= (b^2 - n^2) + (c^2 - m^2) - 2mn \\ &= h^2 + h^2 - 2mn \\ &= 2h^2 - 2mn\end{aligned}$$

$$\begin{aligned}\therefore \cos(\alpha+\beta) &= \frac{h^2}{bc} - \frac{mn}{bc} \\ &= \left(\frac{h}{c}\right).\left(\frac{h}{b}\right) - \left(\frac{m}{c}\right).\left(\frac{n}{b}\right)\end{aligned}$$

Hence $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Model 2

Construct Figure 2 to derive the formulae for $sin(\alpha - \beta)$ and $cos(\alpha - \beta$

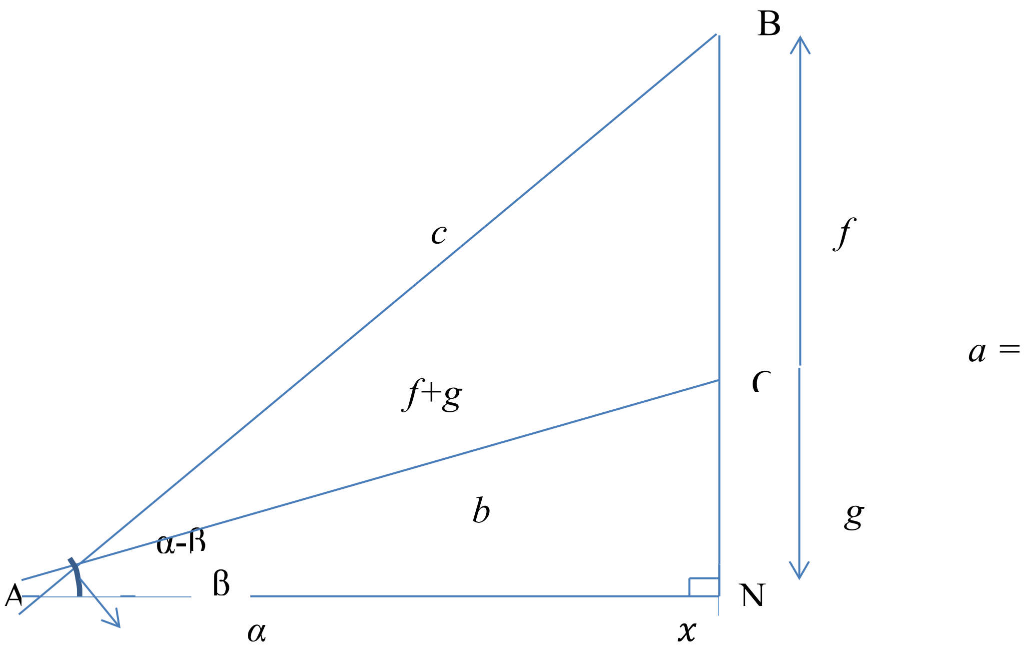

Figure 2

To show:

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Solution:

$$Area\ of\ \Delta ABC = Area\ of\ \Delta ABN - Area\ of\ \Delta ACN$$

$$\Longrightarrow \frac{1}{2}bc\sin(\alpha - \beta) = \frac{1}{2}cx\sin\alpha - \frac{1}{2}bx\sin\beta$$

$$\Longrightarrow \sin(\alpha - \beta) = \sin\alpha . \left(\frac{x}{b}\right) - \left(\frac{x}{c}\right) . \sin\beta$$

$$\text{Hence } \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

To show**:**

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Solution:

Use the same Figure 2:

Applying the Cosine Rule to **Δ**ABC:

$$f^2 = b^2 + c^2 - 2bc\cos(\alpha - \beta)$$

$$\therefore 2bc\cos(\alpha - \beta) = b^2 + c^2 - f^2$$

$$= (x^2 + g^2) + (x^2 + a^2) - f^2$$

$$= 2x^2 + g^2 + (f+g)^2 - f^2$$
$$= 2x^2 + 2g^2 + 2fg$$
$$\Rightarrow bc\cos(\alpha - \beta) = x^2 + g(f+g)$$
$$\Rightarrow \cos(\alpha - \beta) = \frac{x^2}{bc} + \frac{ga}{bc}$$
$$= \left(\frac{x}{c}\right) \cdot \left(\frac{x}{b}\right) + \left(\frac{a}{c}\right) \cdot \left(\frac{g}{b}\right)$$

Hence $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

DISCUSSION

Two different models have been presented to derive the compound formulae for $\sin(\alpha \pm \beta)$ and $\cos(\alpha \pm \beta)$. In the first model, the area of a triangle is identified as the core knowledge where

$$Area\ of\ \Delta = \frac{1}{2} base \times height$$
$$= \frac{1}{2} absin\theta,$$

where θ is the angle between the sides a and *b*.

Thereafter, this concept is expanded to derive the formula for $\sin(\alpha + \beta)$. Similarly, using the Cosine rule as the core concept and applying it to the same model, the compound formula for $\cos(\alpha + \beta)$ is derived. To derive the compound formulae for $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$, similar approaches have been applied to the second model.

It is to be noted that the structures in all the derivations presented in these two models are pyramidal in nature. Mathematical concepts learnt in basic trigonometry are connected in each of the models constructed in order to derive all these formulae. Each of the derivations is done in a single iteration.

In both these models, the complexity is reduced, the proofs are very clear and consolidate earlier trigonometric basic concepts taught. Furthermore, one diagram is needed for deriving both the identities for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ and, another one for $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$. The methods used are all efficient and effective, and enhances quality learning with minimum effort and time once the relevant core knowledge is identified. Hence, this type of approach to teaching must be commended at the high school level.

Furthermore, the proposed methods are flexible and provide room for students to modify or even to rotate the diagrams and come up with their own methods of deriving the proofs. This will give students an opportunity to be in control of their own learning which, is most fundamental in the learning of mathematics. Using these identities and diagrams, learners can further develop their own methods to derive other important trigonometric formulae which are in common use in higher mathematics - such as $\sin k\alpha$, $\cos k\alpha$, $\tan k\alpha$ (k = 2, 3,.. or $k = p/q$, $q > 0$, $k, p, q \in \boldsymbol{R}$).

My approach shows that there is no need for students to start with the proof of $\cos(\alpha - \beta)$ or just to accept it before deriving the other compound formulae. Starting with the sine compound formulae, the derivations are much easier to the students and build up trigonometric concepts smoothly. Students need to be exposed to such proofs at an early age as preparation for higher mathematics later in life. My proposed methods give them the opportunity to appreciate the elegance in the proofs. There is little chance that these proofs will be re-taught at tertiary level. Curriculum designers for CAPS will have to re-think on this section of trigonometry. Book authors should also, in my opinion, revise their proofs. I will leave it to the readers to compare my methods in deriving the compound angle formulae with those presented in the various South African textbooks.

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INVESTIGATING THE ROLE OF DEPARTMENTAL HEADS AS A CRUCIAL LEVER IN EFFECTIVE CURRICULUM DELIVERY IN SOUTH AFRICAN SECONDARY SCHOOLS: THE CASE FOR MATHEMATICS AND PHYSICAL SCIENCE

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The study provided a profile of Departmental Heads (DHs) of Mathematics and Science in South African secondary schools from eight educational districts. In most schools Mathematics and Science were supervised by the same Departmental Head. The profile of these DHs among others revealed the following:

(i) That only 43% of the Departmental Heads were female, reflecting the world-wide concern of poor female participation in Mathematics and Science;

(ii) That 34% of the DHs had either not studied any Mathematics or Science at all, or have done so at significantly lower levels only. Put differently, a significant number of DHs, supervise subjects they had not (adequately) studied.

(iii) That while they are a heterogeneous group they receive limited support from the system, and when they do get support it is a one size fits all intervention.

Crucial policy and practice implications emerge from these findings. Some of these would speak to our DH recruitment practices in Mathematics and Science. Other implications would have a bearing models of professional development, supervision and mentorship for DHs as well as for classroom-based teachers.

INTRODUCTION

In the quest for quality teaching and learning in South African upper secondary school classrooms, the Department of Basic Education (DBE) has foregrounded the role that continuous teacher development and support has to play. The focus depicts the nature and extent of support and mentorship teachers receive within a localized environment at district or school level. In the case of Mathematics and Science, clinical supervision by Departmental Heads (DHs) and subject advisors as a way to support teachers becomes crucial.

While subject advisors provide support and development from outside the classroom, the departmental heads (DHs) are strategically positioned within the school to lead, support and supervise teachers closely and regularly. This paper explored the role of DHs as a crucial lever in curriculum management and delivery.

From within the context of professional development for DHs, the study used a set of research questions to understand whether and how DHs can adequately supervise and support Mathematics and Science classroom-based educators. The age and gender spread of DHs as well as what the DHs have studied and/or teach have been integrally woven into a set of tools to investigate the role of departmental heads in Mathematics and Science.

BACKGROUND AND RATIONALE

The roles of our Departmental Heads, their ways of thinking and judging determine the scope and effectiveness of their transformative influence on the teachers and learners. The curriculum management function of the DHs is in a continuous evolution and improvement. The last few decades have imposed new challenges, debating the role that the DHs should play in supporting and monitoring the performance of teachers in response to the technological advances that have arisen towards the fourth industrial revolution.

The Education 2030, Incheon Declaration (2015:7,8), among the twenty points made (i) inclusion and equity in and through education was perceived as the cornerstone of a transformative education agenda; and (ii) gender equality recognized as important for achieving quality education for all. Further, one target was set that “By 2030, eliminate all gender disparities in education and ensure equal access to all levels of education” (p.21). The UNESCO Report, *Cracking the Code: Girls’ and women’s education in STEM (2017)* documents the status of girls’ and women’s participation, learning achievement and progression in STEM education” (p.16). The latter report indicated that in ten (10) developed countries participation of males far outstrips that of females in both Advanced Mathematics and Advanced Physics in Grade 12.

The National Development Plan (NDP, 2012), a developmental roadmap by the South African government, provides a common focus for actions across all sectors and sections of the South African society to collaborate and make a meaningful contribution towards the attainment of quality education for all. Like the Incheon Declaration, the NDP envisages that by 2030 schools will provide all learners with quality education, especially in Literacy, Mathematics and Science. The NDP vision, however, did not extend far enough to describe the kind of departmental heads as well as support these would give to classroom-based teachers of Mathematics and Science. This study explored the factors surrounding the upper secondary school Mathematics and Science departmental heads that would enhance or impede the support and supervision they would offer to teachers.

Two other tools, *The National Strategy for Mathematics, Science and Technology (MST) Education (2019-2030)* as well as the *Mathematics Teaching and Learning*

Framework for South Africa (2018) also provide strategic thrusts towards improving participation and performance in the STEAM field. The Framework sets out to establish the South African Mathematics teaching and learning identity (p7) by indicating five key objectives that spell out how mathematics should be taught. However, both the MST Strategy and the Mathematics Framework, too, fall short at exploring the departmental heads and the support they would give to Mathematics and Science teachers.

While the teacher in the classroom determines a lot regarding what is taught and how it is taught, the study posits that it is even more pertinent to know the nature of support this teacher receives from his or her supervisor, in this case the departmental head, on a day to day basis. In order, to investigate this, the following research questions were used in the study:

(a) What is the age and gender spread of departmental heads in districts studied
(b) To what extend do the DHs supervise the subjects they have studied for?
(c) How effective and influential do DHs perceive their own roles within a school?

The research explored the generally complex terrain of departmental heads in a school, with their fundamental responsibility to guarantee the necessary support to teachers in a way that contributes to the improvement of performance. It is pertinent, therefore, to systematize the educational theories and practices that inform foundational pedagogical choices related to the functions of the DHs in a way that seeks to enhance the quality of teaching and learning.

THEORETICAL UNDERPINNINGS

Over the years several theories have been developed for the management and leadership of processes, some of the most outstanding reflect the need for direct work with people as the center of any process. The behavioral sciences played a strong role in helping to understand how the needs of workers and those of the organization could be better aligned. With the Human Relations movement, training programs recognized the need to cultivate supervisory skills, e.g., delegating, career development, motivating, coaching, mentoring, etc.

Systems Theory

Systems theory is a way of elaborating increasingly complex systems across a continuum that encompasses the person-in-environment (Anderson, Cater and Love, 1999). Systems theory also enables us to understand the components and dynamics of client systems so that one can interpret problems for appropriate and balanced interventions. This will enhance “goodness of fit” between individuals and their environments (Hargreaves, 1992). The theory has brought a new perspective for managers to interpret patterns and events in the workplace. They recognize the various parts of the organization, and, in particular, the interrelations of the parts, the coordination of central administration with its programs, engineering with manufacturing, supervisors with workers, etc. Hargreaves (1992:10) also describes an ecological systems model which adapted to education would be an individual DH, in a family (school) which is also located in the family of origin (district) and finally accounting to the community (the department). This provides a crucial framework for understanding the work of departmental heads within the embeddedness of the education system.

Collegiality and Clinical Supervision

In education, systematic coordination takes a leaf from Systems Theory. Little (1987) argued that 'by working closely with colleagues teachers derive instructional range, depth and flexibility'. The fostering of a strong school culture of interdependent collegiality introduces an openness that enhances mentorship and support among teachers. Grimmet and Crehan (in Fullan and Hargreaves, (1992:63) argue that "such openness to observing, to being observed, and to discussing observed classroom practice tends to breakdown isolationist barriers and promote the norm of collegial interdependence".

For departmental heads, who are subject supervisors in a school context, to achieve such collegial atmosphere they need to be confident and competent in the subjects they supervise. They need to have studied these subjects in greater detail.

Grimmet and Crehan further suggested a particular type of collegiality called *organizationally-induced* collegiality. This type "is characterized by 'top-down' attempts at fostering 'bottom-up' problem-solving approaches to school improvement through careful manipulation of …the environment within which teachers live and work and have their professional being" (p70). This is a powerful construct because departmental heads who are confident and open can begin to initiate discussion among teachers in a bottom-up fashion towards higher forms of engagement and cognitive attainment.

But how was the study conducted? It is important to describe how the study on Mathematics and Science departmental heads was designed and analyzed.

METHODOLOGY

While DHs were on professional development workshops to enhance the roles and responsibilities of Departmental heads in curriculum management, they were requested to complete a questionnaire that explored their profile. In addition, focused group interviews were conducted with groups of about six participants. Completing the questionnaire and going through the interview complemented each other and provided a more comprehensive picture but were not dependent on each other. In this case it did not matter whether one was done before the other.

While eight districts completed the questionnaires, focused group interviews were conducted at three districts in North West (6 participants), Limpopo (5 participants) and Mpumalanga (5). The same set of interview questions were used with all focused groups, although the flow of responses was fluid depending on the groups being interviewed. The participants in the groups were randomly selected. Among others, the focused interview explored the role and influence of departmental heads in a school. The interview was audio recorded.

FINDINGS

The eight districts that participated in the study come from five provinces. Five of the districts tended to be more rural and generally performing lower than others. School conditions and factors in these districts were relatively difficult, and teachers generally preferred to teach in more urban settings where conditions are better. There were teachers who are foreign nationals in these districts but very rarely attended the departmental head professional development since they are generally not appointed as departmental heads.

The spread of the departmental heads according to their *age* and *gender* in the sample were of special educational interest.

Age and Gender spread of departmental heads

The age and gender spread of departmental heads is shown in Table and Table 2 below. Interesting patterns emerge from the data that would begin to inform both policy and practice. The nature and content of professional development programmes should be guided by the emerging patterns, for instance what kind of programme would be appropriate for a twenty five year old with three years' experience who is in the same programme as a fifty five year old with thirty five years' experience.

Table 1: Age distribution of Departmental Heads

		AGE					TOTAL
District	Province	25-34	35-44	45-49	50-54	55 - & Above	
Alfred Nzo (Maluti area)	EC	9	22	14	10	3	58
Amathole East (Butterworth area)	EC	9	15	3	2	0	29
Dr RSM (Vryburg area)	NW	14	16	12	5	9	56
Francis Baard (Kimberley area)	NC	10	10	11	7	4	42

Gert Sibande (Ermelo area)	MP	7	20	19	20	3	69
Mogalakwena (Mokopane area)	LP	3	9	15	25	12	64
NM Molema (Mafikeng area)	NW	36	22	25	12	7	102
ORT Coastal (Mthatha area)	EC	4	7	2	1	1	15
TOTAL		92	121	101	82	39	435

- *The area of the district is given in brackets to assist the reader as to the location of the district. It should, however, be noted that the district is much bigger than the area indicated.*

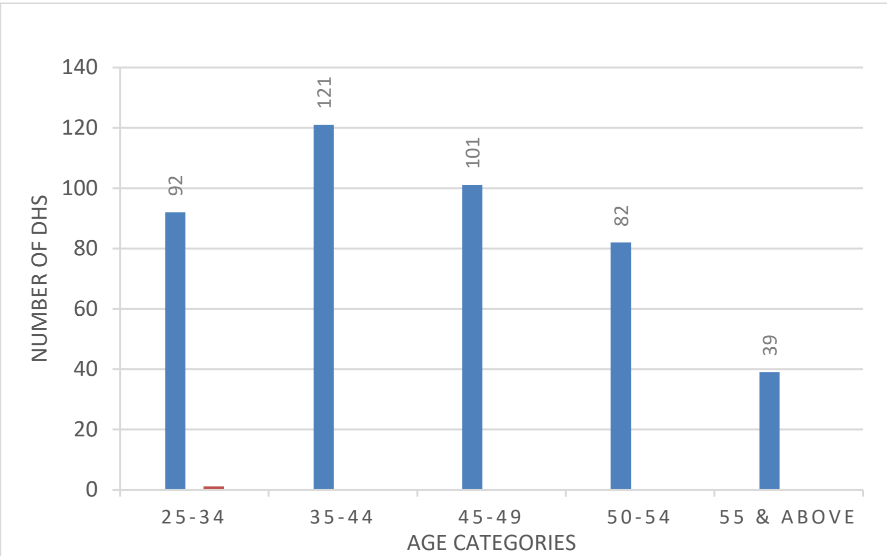

Figure 1: N=435

A fair number of departmental heads is youthful: 49 % below the age of 44, of which 21% are under 35 of age. Appropriate programmes to capacitate these would be well invested. On the other hand, 9% of the DHs were 55 years and older. Certainly a differentiated professional development approach is necessary for departmental heads.

The Gender Spread

It was of our keen research interest to investigate the gender spread of the departmental heads that participated in the professional development. The UNESCO report (2017) on "*Cracking the Code: Girls' and womens' education in science, technology, engineering and mathematics (STEM)*" lifted a number of factors that enhance or impede girls and women participation in the STEM field. It was crucial to check how the South African context reflects these international factors. The UNESCO report indicates that "teachers' sex is also an influential factor as female teachers can serve as role models for girls" (p.50). Proposing plausible interventions, the report says:

> "Mentorship programmes have been found to improve girls' and womens' participation and confidence in STEM studies and careers. A US study found that, at lower secondary level, girls who were mentored by female role models during summer activities showed greater interest in science and mathematics when introduced to potential STEM career opportunities" (p.69).

Table 2: Gender Spread of the departmental heads in 8 districts in South Africa

District	Male	Female	Total	% female
Alfred Nzo	28	30	58	51.7
Amathole East	12	17	29	58.6
Dr RSM	35	21	56	37.5
Francis Baard	18	24	42	57
Gert Sibande	45	24	69	34.8
Mogalakwena	40	24	64	37.5
Ngaka Modiri Molema	64	38	102	37.2
OR Tambo Coastal	7	8	15	53.3
TOTAL	249	186	435	42.8

While the Table gives the spread across the eight district, the graph below shows overall spread of male and female departmental heads. In this regard, a picture seemed to emerge indicated that districts that tended to be more rural were having fewer female departmental heads.

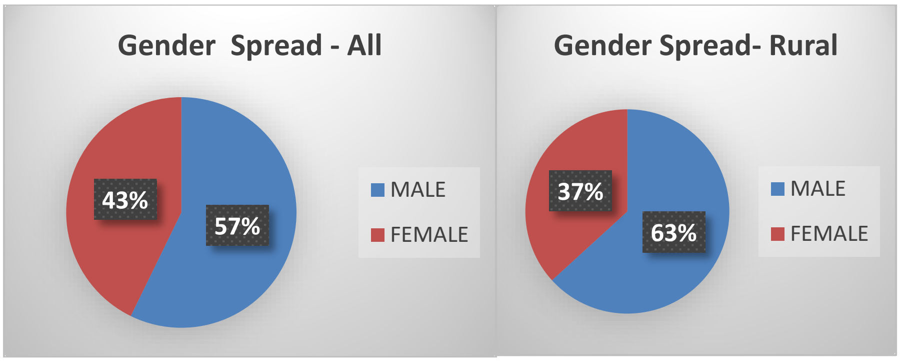

Figure 2

- Both Table 2 and Figure 2 above indicates that 42.8% of the departmental heads was made up of females. However, in four of the eight districts (Dr RSM, Gert Sibande, Ngaka Modiri Molema and Mogalakwena) which are more rural, the average female percentage is 36.8%. The other districts tended to be relatively urban and further, in OR Tambo Coastal, attendance was very low, and not representative of the district.
- The figure of 36.8% would be closer to reflecting the actual female population among the departmental heads in a secondary schools in South Africa.

Departmental Heads' work in relation to what they teach, supervise and have studied for

In order to understand the effectiveness of departmental heads in leading Mathematics and Physical Science in the upper secondary schools, it was crucial to investigate the their roles and responsibilities in relation to *what they teach, what they supervise* as well as *what they have studied for*. Tables 3 & 4 below provide insightful figures in this regard.

Table 3: What DHs Teach and Supervise, N=437

District	Teaching FET Mathematics and Physical Science	Teaching Math Lit, GET mathematics and Natural Science	Teaching other
Alfred Nzo	32	9	17
Amathole East	21	3	5
Dr RSM	35	7	6

Francis Baard	30	8	3
Gert Sibande	41	15	6
Mogalakwena	47	18	8
Ngaka M. Molema	72	16	14
OR Tambo Coastal	9	5	1
TOTAL	**287**	**81**	**60**

- Sixty of 435 participants (13.8%) did not teach any Mathematics and Science, although they supervise them. This factor indicated the seriousness of the situation in Mathematics and Science. In a sense, no value-adding supervision can these individuals provide to super the teachers.
- Eighty one of the 435 participants (18.6%) taught Mathematics and Science in the GET Phase (i.e., at lower secondary school level, (Grade 8 & 9), while they supervise these subjects at upper secondary school level. The probability of these DHS to meaningfully support the FET Mathematics and Science teachers is very minimal, if any. Combined, 32.4% of the participants do not have adequate knowledge to supervise or mentor, let alone lead school-based professional learning communities.
- Two hundred and eighty seven of the 435 participants (66%) teach and supervise Mathematics and Science upper secondary school level.

Table 4: DHs Qualifications and Major Subjects

DISTRICT	Diploma	One Degree	2 0R 3 Degrees	Majoring in Maths and/or Science	Other Majors
Alfred Nzo	33	24	1	42	16
Amathole East	18	9	0	19	8
Dr RSM	13	36	7	50	6
Francis Baard	9	22	3	36	5
Gert Sibande	33	21	1	54	12
Mogalakwena	42	20	2	47	12
Ngaka Modiri Molema	28	45	29	86	16
OR Tambo Coastal	3	8	2	9	4
Totals	**179**	**185**	**45**	**343**	**79**

- All DHs in the sample were professionally qualified. The fact that all departmental heads in the study are professionally qualified is appreciated. However, this fact hides the issue of suitability of the DHs to support teachers teaching Mathematics and Science.
- Seventy nine of the 435 participants (18.2%) have majored in subjects other than Mathematics and Science. In this category we find some who studied English, History, Agricultural Science, etc. This means these DHs, despite supervising Mathematics and Science, have not studied these subjects beyond Grade 12.
- 78.2 have majored in Mathematics and Science
- 3.6% have not indicated what they majored in at college or university

Effective and Influential? Views by Departmental Heads themselves

A set of five sub-questions formed the thrust of the interview but due to space, responses to only three are reflected on here. The questions and some responses to them are captured hereunder.

(i) How effective are you as a departmental head?

- ✓ *I feel I am effective. I brought in some innovations when I came in as a departmental head. I encouraged teachers to work together, and learners to work together. This has been well received. Our results are starting to improve.*
- ✓ *For me, not really. The push for matric results forces us to concentrate mainly on Grade 12 and somehow neglect the lower grades. This is such a problem. We are aware of it but we don't seem to be getting it solved.*
- ✓ *I think I'm doing well. In my case, the subject advisors visit us regularly and this helps us a lot. They are very helpful, sometimes they bring us some resources.*

(ii) How do you rate your voice on influence as a member of the School Management Team?

- ✓ *I do a lot other activities like school time table, examination time table as well as NSC results analysis. And because of this the principal works very closely with me. This gives me more influence some decisions.[This view was expressed by a number of the participants across the three groups].*
- ✓ *Sometimes when I push for more time and resources for my department, I meet up with counter views that say 'all subjects are the same, why prioritize Mathematics and Science?' This does not augur well for these subjects.*

(iii) What challenges do you face as a departmental head?

- ✓ *My school did not have a Mathematics and Science for a long time, and when I was appointed I was the youngest among the older and more* *experienced in the SMT. This posed a challenge to have my voice and views taken seriously enough.*

- ✓ *I majored in Life science in my studies and when appointed a Mathematics and Science departmental heads, there were challenges of adequately supervising and supporting subjects like Mathematics and Physical Science. However, I established small teams within each subject (say, Mathematics teachers) so that they can assist one another with content.*
- ✓ *Although I have done both Mathematics and Chemistry in my studies, I struggle to do practicals because we do not have a functional laboratory. Much of the teaching in science is done from the textbook.*

IMPLICATIONS FROM THESE FINDINGS

Crucial implications for policy and the nature professional development emerge from these findings:

(a) Given the wide **age spread** of the DH participants purposeful and strategic decisions could be made about nature and content of professional development programmes:

- ✓ Professional development could be **differentiated**, so that the new entrants to the DH field could be given adequate induction. This will prepare them for the huge task of curriculum management and supervision.
- ✓ Experienced DHs could be used to mentor for the new ones
- ✓ More investment could be placed on those relatively younger (younger than 44 years) as compared to those who are about to exit the system (55 years and older).

(b) The **gender spread** of the DHs could offer crucial implications for development and recruitment strategies:

- ✓ Deliberately recruit female DHs into the management levels. This can be turned into an active policy prescript.
- ✓ Create a supportive and conducive environment for female DHs to function optimally in a school
- ✓ Not only is meaningful presence of the female departmental heads at the leadership and management level important, it will also encourage the female students to do well in Mathematics and Science.

(c) About **responsibility and specialization**

Two key findings in this regard were (i) that some departmental heads did not teach the subjects that they supervised (13.8%), and that (ii) other departmental heads taught Mathematics and Science at grades lower than what they supervised (18.6%). This poses a challenge in that support, mentorship and oversight of Mathematics and Science in secondary schools tended to be compromised. By inference, capacity to provide quality guidance and supervision was limited. Once again, appropriately differentiated professional development programmes would need to be designed and implemented to help support the departmental heads of these varying capacities.

(d) About **effectiveness and influence**

It was clear from the three focused groups that while the schools context (resources and culture) are different, most departmental heads felt effective and confident in their roles. This is a crucial factor that needs to be harnessed even further because confident and supported departmental heads have potential to raise performance the STEAM subjects.

Further, a differentiated professional development for departmental heads can be more effective. New and younger departmental heads' needs can be packaged into an appropriate programme for development and support. Obviously, another (different) tailor-made programme would be needed for those who have not studied these subjects at all.

CONCLUSION

Although the study was small and not representative, it however elicited crucial findings that would inform both policy and practice in terms departmental heads supervision, support and development of Mathematics and Science teachers in South African secondary schools. Further, differentiated professional development for departmental heads or teachers would be more appropriate for supporting these critical disciplines in schools.

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INTERROGATING LEARNERS' APPROACHES TO SOLVING QUADRATIC EQUATIONS AND INEQUALITIES

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There is always concern about learner performance in Mathematics in South Africa and other countries. Despite the improvement in grade 12 Mathematics passes in South Africa in the past five years, there are still a number of schools where learner performance in Mathematics is very poor. The author, of this paper, was involved in a supplementary tuition project for Mathematics at one of those schools. In this project, grade 12 mathematics learners were given a series of two hour lessons on key content/topics, followed by tests. One of the tests was on quadratic equations and inequalities. In this paper, learners' approaches to these test questions are analyzed with a view to determining their levels of understanding and reasoning. Some recommendations are made in this regard.

INTRODUCTION AND BACKGROUND

Mathematics is an important subject in South Africa and other countries. In South Africa, at the beginning of grade 10, learners have to choose between Mathematics or Mathematical Literacy. Each school has its own criteria for selection into Mathematics or Mathematical Literacy. Learners who usually pass Mathematics in grade 9 are given the option to do Mathematics in grade 10. However, it would appear that there is a big gap between Mathematics in grade 9 and Mathematics in grade 10. According to some grade 10 teachers, in personal communication with the author, learners who just passed grade 9 Mathematics tend to struggle in Mathematics in grades 10 and 11. These learners then change to Mathematical Literacy. Table 1 shows the national percentage pass rates for Mathematics and Mathematical Literacy for the past 5 years:

Table 1 Grade 12 Percentage pass rates in Mathematics and Mathematical Literacy enrolment (2014 – 2018)

Year	Mathematics	Mathematical Literacy
2014	53,5%	84,1%
2015	49,1%	71.4%
2016	51,1%	71,3%
2017	51,9%	73,9%
2018	58,0%	72,5%

(DBE, 2018)

We note that after being stable for four years, the Mathematics results improved appreciably in 2018. At the same time, after reaching a peak of 84,1% in 2014, the Mathematical Literacy results dropped in 2015 and have since stabilized.. To put the 2018 Mathematics results in context, one may look at the views expressed by AMESA on the 2018 Mathematics Papers.

Mathematics Paper 1 was described as *"A well-balanced, fair and learner friendly paper with a few very challenging, unfamiliar, non-routine and unexpected questions which will definitely separate the top learners from the rest"* (AMESA, 2018:5).

The overall verdict for Mathematics Paper 2 was *"A well-balanced but very challenging paper"*(AMESA, 2018:11).

It is possible that the improved Mathematics results in 2018 could be attributed to improved teaching practices at schools as well as the various winter and spring schools held by districts and provinces for learners from underperforming and other schools.

The author was approached in July 2018 to assist grade 12 learners from a high school with supplementary tuition in Mathematics. All learners had failed Mathematics in June and the school principal and school governing body were desperate for this support. The author agreed and assisted the learners with supplementary tuition comprising 11 two hour lessons based on key grade 12 topics/sections. At the beginning of each lesson, learners wrote a half-hour test which covered the content of the previous lesson. The very first lesson in this supplementary tuition programme was on quadratic equations and inequalities. Learners wrote a test on this section of the work at the beginning of the second lesson. This paper interrogates learner performance in this test.

LITERATURE REVIEW

The literature review for this study shows work done by some researchers on students' approaches to solving quadratic equations in different contexts and the type of reasoning and understanding they displayed.

A study by Vaiyavutjamai, Ellerton and Clements (2005) on how students solve two straightforward quadratic equations in three countries (Brunei, Thailand and the United States) revealed confusion on the part of students with respect to the concept of a variable as seen in the quadratic equations, $x^2 = 9$ and $(x - 3)(x - 5) = 0$. They found several misconceptions regarding variables which were obstacles to understanding quadratic equations. For example, students thought that the two x's in the equation $(x - 3)(x - 5) = 0$ stood for different variables, even though most of them obtained the correct solutions $x=3$ or $x=5$. They concluded that students' performance in this context reflected rote learning and a lack of relational understanding.

Didiş, Baş & Erbaş's (2011) study examined grade 10 students' procedures for solving quadratic equations with one unknown, using an open ended test. The results reveal that students attempted to solve the quadratic equations as quickly as possible without paying much attention to their structures and conceptual meaning. They report, further, that students' written answers provided clues to their reasoning when solving quadratic equations, concluding that the reasoning was based on instrumental understanding. They found that factorizing quadratic equations was challenging for students when presented in non-standard forms and structures. They recommended that teachers introduce various kinds of quadratic equations in different structures rather than just in the standard form. They state that it would also be helpful for students to understand the factorization techniques as relational when teachers clearly emphasize meaning of the null factor rather than presenting it just as rule.

A paper by Makgakga (2014) outlines the errors made by learners when solving quadratic equations. This was divided into two components, types of errors and misconceptions, as well as diagnosing errors and misconceptions. Makgakga also lists some reasons which contributed to the difficulties faced by learners and concludes by stating that "*teachers are not supposed to just identify learners' errors and misconceptions but they should also diagnose those errors in order to understand learners' challenges*" (Makgakga, 2014: 290)

Govender (2015) discusses the importance of quadratics (quadratic equations and functions) in the South African mathematics curriculum. These topics form a significant part of the grade 12 Mathematics examinations. Govender states that teachers' preparedness, content knowledge and sound pedagogical approaches are key factors when teaching quadratics and other topics. This would ensure that children would learn mathematics in a more structured and coordinated manner. This will also enable learners to do the following:

- Make connections between various components of the curriculum,
- Use their knowledge and ability in quadratics (from algebra) in another section of mathematics (for example, trigonometry)
- Work with linear, quadratic and cubic equations and functions and note any similarities and differences
- Apply their knowledge of quadratics equations and functions in other subjects such as Physical Sciences (Govender, 2015)

Researchers, in this literature review, have highlighted certain issues in the teaching and learning of quadratic equations and attempt to provide solutions. This literature review is now concluded with the following summary:

- Learners tend to use instrumental reasoning, rather than relational understanding when solving quadratic equations

- Learners struggle with quadratic equations when presented in non-standard forms and structures
- Errors made by learners should be diagnosed in order to understand their challenges when solving quadratic equations
- Quadratics form an important part of the South African Mathematics curriculum and the teaching of this important section of the work should be done in an integrated manner to impact positively on learner understanding of the various concepts involved

RESEARCH QUESTION

As stated in the above summary quadratics is an important part of the South African mathematics curriculum. It also features prominently in the grade 12 Mathematics examination papers (Govender. 2015). Thus, if learners are to do well in grade 12 mathematics examinations, then it is imperative that they score well in quadratic equations and functions.

This means that more should be known about how learners approach these types of questions and the type of reasoning they display. It was with this in mind that the following research question was posed for this study:

"What inferences may be drawn when some learners work through questions involving quadratic equations and inequalities in a test?"

To answer the research question, the following subsidiary questions were formulated within the context of the research question:

- How do learners perform in questions on quadratic equations and inequalities in a test?
- What understanding is displayed by learners when working through questions on quadratic equations and inequalities?

RESEARCH SAMPLE

The sample for this study was made up of 20 grade 12 Mathematics learners who attended school in one of the Black townships of the Eastern Cape. The learners were transported to a central venue where they were taught by the author. All learners participated voluntarily in this study.

THEORETICAL FRAMEWORK

This research focuses on how learners perform in questions on quadratic equations and inequalities and the type of understanding shown by the learners when working through these questions. Skemp (1976) speaks about teaching mathematics for, what he referred to as to as, *relational* rather than *instrumental* understanding. He states that, while instrumental understanding may have short-term benefits, such as

taking less time to learn skills and procedures, relational understanding is more adaptable to new tasks, is easier to understand, and is an appropriate goal in itself.
Building on from Skemp, Kilpatrick, Swafford and Findell (2001) propose five “intertwining strands” of mathematical proficiency, namely:

- *Conceptual Understanding:* comprehension of mathematical concepts, operations and relations;
- *Procedural Fluency:* skill in carrying out procedures flexibly, accurately, efficiently and appropriately;
- *Strategic Competence:* ability to formulate, represent, and solve mathematical problems;
- *Adaptive Reasoning:* capacity for logical thought, reflection, explanation and justification and
- *Productive Disposition:* habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy.

Kilpatrick et al. (2001:116) state that these strands are not independent but instead, “represent different aspects of a complex whole”, thus leading to the notion of intertwined strands. Notwithstanding the importance of all the strands, for purposes of this study, two of the strands mentioned, “conceptual understanding” and “procedural understanding”, are addressed.
One may infer, from the preceding descriptions, that instrumental understanding is aligned with procedural fluency while conceptual understanding incorporates relational understanding. Since this study interrogates the reasoning and understanding displayed by learners when working through questions on quadratic equations and inequalities, an appropriate framework in which to locate this study is *“learner understanding of quadratic equations and inequalities within the context of a mathematics assessment task”*. This study also involves both a description and an interpretation of this learner understanding, thus locating the study within the interpretive research paradigm (Bassey, 1999).

RESEARCH METHODOLOGY

This study involved the collection of both qualitative and quantitative data (Cohen and Manion, 1985). Qualitative data included the lesson notes used and the learner responses to the test questions, and analyses thereof. Government documents and learners’ tests performances provided quantitative data.

DATA COLLECTION

The following data sources were used for this study:

- Government documents such as the Curriculum and Assessment Policy Statement (CAPS) for Mathematics (FET), the assessment guidelines and past year grade 12 Mathematics Papers
- Lesson notes used by the author when teaching the section on quadratic equations and quadratic inequalities

- The test set by the author
- Learner performance in the test
- Learners' test question papers showing how learners attempted the questions

DATA

TOPICS FROM THE CAPS DOCUMENT

The CAPS document shows that the topics, relevant to this study, exponents and surds, quadratic equations and inequalities, are done in term 1 of grade 11 (DBE, 2011a:30). Some questions are given alongside these topics for clarification. The levels of difficulty of each question are also indicated.

MATHEMATICS EXAMINATION GUIDELINES

The examination guidelines for Mathematics (DBE, 2017) show that algebra, equations and inequalities are assessed in Grade 12 Mathematics Paper 1. In terms of alignment to the CAPS document, the term algebra (rather than exponents and surds) is used in the examination guidelines as it appears to be more inclusive.

QUESTIONS FROM PAST YEAR PAPERS

Questions on algebra, equations and inequalities are likely to be included as the first question in grade 12 Mathematics Paper 1. In this regard, the first questions of the last two end of year Grade 12 Mathematics Paper 1 are shown next:

October/November 2017: Question 1

1.1 Solve for x

1.1.1 $x^2+9x+14=0$ (3)

1.1.2 $4x^2+9x-3=0$ (correct to 2 decimal places) (3)

1.1.4 $\sqrt{x^2-5}=2\sqrt{x}$ (4)

1.2 Solve simultaneously for x and y

$3x-y=4$ and $x^2+2xy-y^2=-2$ (6)

1.3 Given $f(x)=x^2+8x+16$ (4)

1.3.1 Solve for x if $f(x)>0$

1.3.2 For which values of p will $f(x)=p$ have TWO unequal negative roots? **(4)**

[24]

(DBE, 2017a)

October/November 2018: Question 1

1.1 Solve for *x:*

1.1.1 $x^2 - 4x + 3 = 0$ (3)

1.1.2 $5x^2 - 5x + 1 = 0$ (correct to 2 decimal places) (3)

1.1.3 $x^2 - 3x - 10 > 0$ (3)

1.1.4 $3\sqrt{x} = x - 4$ (4)

1.2 Solve simultaneously for x and y

$3x - y = 2$ and $2y + 9x^2 = -0$ (6)

1.3 If $3^{9x} = 64$ and $5^{\sqrt{p}} = 64$ calculate without the use of a calculator (4)

the value of $\dfrac{\left[3^{x-1}\right]^3}{\left(\sqrt{5}\right)^{\sqrt{p}}}$

[23]

(DBE, 2018a)

The following may be noted from these questions:

- The assessment of quadratic equations involves the use of factorisation and the quadratic formula
- Learners need to know how to solve quadratic inequalities. However, the 2017 paper also brings the function notation $f(x)$ into the picture.
- Solution of simultaneous equations appear in both questions
- Both questions on exponents and surds eventually lead to solution of quadratic equations which factorise. However, the solutions should be checked by the learner
- Question 1.3.2 (from 2017) and question 1.3 (from 2018) are set at a slightly higher level

AUTHOR'S LESSON NOTES

The author constructed notes on the following sections:

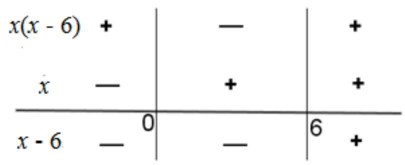
- Solving quadratic equations using factorisation and the use of the quadratic formula
- Solving quadratic inequalities
- Solving two equations (one linear and one quadratic) simultaneously

- Working with simple exponents and surds
- Solving problems involving quadratic equations, exponents and surds and other sections of the curriculum

The author covered a number of questions during the contact session. Learners were shown various approaches to solving the questions and then given additional questions to consolidate their learning. For the work on quadratic inequalities, learners were shown both the number line method and the graphical method.

For example, learners were given the following question to solve: $x^2 - 6x > 0$ The left hand side was factorized to get: $x(x-6) > 0$. The critical values were obtained by equating factors x and $x - 6$ to 0.

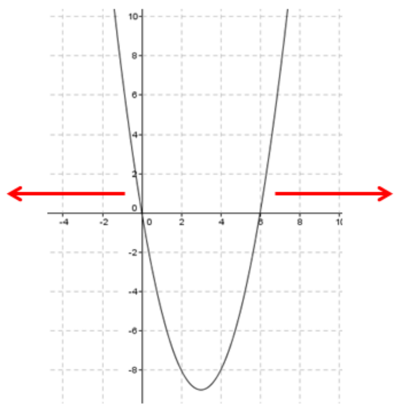
Critical values: $x = 0$ and $x = 6$. The appropriate number line is drawn below:



The solution is located where $x(x-6)$ is positive; to the left of 0 and to the right of 6. Thus, the solution is written as follows:

$x(x-6) > 0 \Rightarrow x < 0$ or $x > 6$

The solution may also be shown using a graph. The critical values are the x-intercepts of the graph and we look for values above the x-axis (where the graph is positive, that is, $y > 0$). The two arrows show where the solution is located (to the left of 0 and to the right of 6).



Thus, $x < 0$ or $x > 6$

THE TEST WRITTEN BY LEARNERS

At the beginning of lesson 2, learners wrote a test on content covered in the first lesson. The author set this test, keeping in mind, the structure of Mathematics Paper 1 Question 1.

This test, with mark allocation, is shown below:

Test 1

Solve for *x*: (in numbers 1 – 4)

1. $(2x-3)(x+1)=0$ (2)
2. $x(x-5)=3-x^2$ (3)
3. $2x^2-6x-9=0$ (correct to 2 decimal places) (3)
4. $x^2-3x-4\leq 0$ (3)
5. Solve simultaneously for x and y

   $x^2-xy+2y^2-7=0$ (6)

   $x=y+1$

6 Simplify $\left(\sqrt[3]{\sqrt{65}+1}\right)\left(\sqrt[3]{\sqrt{65}-1}\right)$ (3)

[20]

The author constructed items for the test with a view to ensuring that the test was fair and cognitively balanced. The author's classification of the cognitive level for each question is shown in table 2.

Table 2 Cognitive levels for test

Question	Knowledge (Level 1)	Routine procedures (Level 2)	Complex procedures (Level 3)	Problem solving (Level 4)
1	2			
2	1	2		
3		3		
4	1		2	
5		6		
6				3
Total	4	11	2	3

15 out of 20 marks of the test (75%) consisted of level 1 and level 2 questions. It would be reasonable to claim that the test should be within the grasp of the learners. Learners were given codes A, B, C, ..., T. Their marks, for each question of the test, are shown in table 3.

Table 3 Learner performance in test

Question	**1**	**2**	**3**	**4**	**5**	**6**	**Total**
Learner	**2 marks**	**3 marks**	**3 marks**	**3 marks**	**6 marks**	**3 marks**	**20 marks**
A	1	0	3	1	1	0	6
B	0	1	3	0	0	0	4
C	1	2	3	1	1	1	9
D	2	3	3	3	5	3	19
E	2	1	1	1	1	0	6
F	2	2	3	2	2	1	12
G	0	0	0	0	0	0	0
H	2	1	3	3	6	0	15
I	2	1	3	3	2	0	11
J	2	0	1	1	2	0	7
K	2	2	3	2	2	0	11
L	0	0	1	1	0	0	2
M	2	3	3	2	1	3	14
N	2	0	3	2	0	3	10
O	2	0	1	2	1	3	9
P	0	0	1	2	1	0	4
Q	2	1	1	0	1	3	8
R	2	1	3	1	1	3	11
S	2	3	3	2	6	1	17
T	2	3	2	2	2	0	11
Number correct (full marks)	**14**	**4**	**12**	**3**	**2**	**6**	**9.3 (Average)**

TRENDS FROM TABLE 3

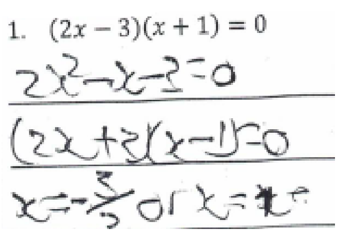
- Despite the quadratic equation being given in the factorised form $a \times b = 0$ only 14 out of the 20 learners got this answer completely correct. Interestingly, four obtained zero for this question
- Question 2 was not given in standard form. This probably resulted in only four learners get full marks for this question. The majority of the learners (14) obtained either 0 or 1 for this question
- Question 3 was given in standard form and 12 out of the 20 learners obtained full marks for this question. This is probably due to the hint that the solutions should be given to two decimal places. Learners know that when this hint is given, they should use the quadratic formula
- Learners struggled with the quadratic inequality (question 4) with only three getting full marks in the question. Those who got 1 or 2 marks for the question were able to factorise the left hand side of the inequality but were not able to go further with the question
- Despite being given a straight-forward question on simultaneous equations (question 5), two learners achieved the full 6 marks and another learner got 5 marks. Unfortunately, all the other learners obtained 2 marks or less for this question
- 6 learners obtained full marks for question 6. The rest obtained either 0 or 1 mark for this question

INSIGHT INTO LEARNERS' SOLUTIONS

For purposes of this section of the data, the author has chosen questions from eight learners tests, based on their responses. There are some analyses and interpretation of each solution.

Question 1

For question 1, learner C and M's solutions were chosen. Learner C attempted the question as follows:



1. $(2x-3)(x+1)=0$

$2x^2-x-3=0$

$(2x+3)(x-1)=0$

$x=-\frac{3}{2}$ or $x=1$

Learner M gave the following solution:

1. $(2x-3)(x+1)=0$

$2x^2+2x-3x-3=0$

$2x^2-x-3=0$

$(2x-3)(x+1)=0$ $\quad\therefore x=\frac{3}{2}$ or $x=-1$

Both learners were not able to move from the factorized form directly to the solutions. They multiplied out and then factorized. Learner C obtained different factors resulting in incorrect solutions. Learner M was able to get the correct factors and the correct solutions. There is no doubt that both learners were not able to make sense of what was given. They saw two given factors, written in brackets, and decided to multiply these factors, possibly using what they learnt in grade 9 and 10, on the multiplication of binomials. These learners applied a procedure, without understanding the concepts involved. Although it worked out for learner M, who got back the original factors, he did not realize that he gave himself extra work.

Question 2

Learners' E and F responses were chosen for this question. Learner E attempted this question as follows:

2. $x(x-5)=3-x^2$

$x^2-5x=3x^2$

$x^2-5x-3+x^2=0$

$=x^2-5x-3=0$ $\quad\therefore x=-4$ or $x=1$

$(x+4)\quad(x-1)=0$

Learner F's response:

2. $x(x-5)=3-x^2$

$x(x-5)-3+x^2=0$

$x^2-5x-3+x^2=0$

$2x^2-5x-3=0$

$(2x+3)(x+\frac{1}{3})$ $\quad\therefore x=+3$ or $x=-\frac{1}{3}$

This question involved solution of a quadratic equation which was given in a non-standard form. Learner E was able to multiply out but did not get the correct standard form of the equation. Learner E did not realize that $x^2+x^2=2x^2$ and, thus, worked with the incorrect quadratic equation.

The incorrect equation obtained, $x^2 - 5x - 3 = 0$, does not factorize. However, the thinking displayed by the learner in getting the factorized form $(x-4)(x+1) = 0$ is difficult to understand as the factors do not multiply out to get $x^2 - 5x - 3 = 0$ While, learner E did not check his/her working, it is also clear that s/he did not display any conceptual understanding .

Learner F showed a good understanding of the procedures involved until it came to factorizing his correct equation $2x^2 - 5x - 3 = 0$. Learner F did not multiply his factors to check whether these were correct. Further, one of his factors was $\left(x + \frac{1}{3}\right)$ and this should have been checked as learners are not usually taught to write a factor in this manner. Both learners knew the procedure to use in this question but errors crept into their workings which were not checked.

Question 3

Performance in question 3 was good as only one learner failed to comprehend the question and obtained 0. 12 learners got full marks for this question. This was probably due to hint given in question. Learners are usually made aware by their teachers that "correct to 2 decimal places" would indicate that the quadratic equation does not factorize and that the quadratic formula should be used.

Learners P and Q's responses were chosen. Learner P responded as follows:

3. $2x^2 - 6x - 9 = 0$ (to 2 decimal places)

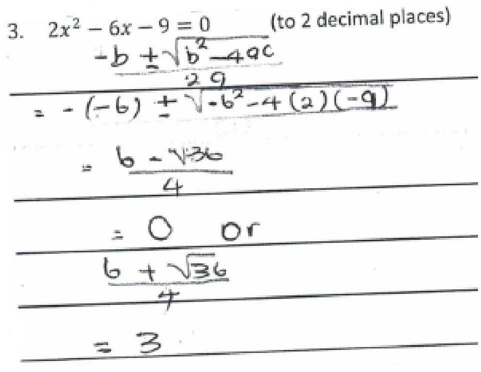
$$\frac{-b \pm \sqrt{(b)^2 - 4(a)(c)}}{2(a)}$$

$$= \frac{-(6) - (-6)\sqrt{(6)^2 - 4(2)(-9)}}{2(2)}$$

$$= \frac{6 \pm \sqrt{78}}{4}$$

$\therefore x = 3.70$ or $x = 0.70$

Learner Q's response:



3. $2x^2 - 6x - 9 = 0$ (to 2 decimal places)

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= -(-6) \pm \sqrt{-6^2 - 4(2)(-9)}$

$= \frac{6 - \sqrt{36}}{4}$

$= 0$ or

$\frac{6 + \sqrt{36}}{4}$

$= 3$

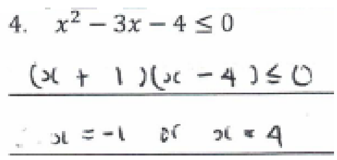
Learner P selected the appropriate formula from the data sheet and was able to substitute into the formula. However, his calculation of $\sqrt{b^2 - 4ac}$ was incorrect, not squaring b = -6 to get an answer of $\sqrt{78}$ which one may classify as an "elementary" error.

Learner Q also obtained an incorrect value for $\sqrt{b^2 - 4ac}$, that of $\sqrt{36}$. In this case, the learner substituted for b as follows: $-(6)^2$ instead of $(-6)^2$. This could also be classified as an "elementary" error.

In both these instances, it would seem that these learners were aware of the procedures involved and their understanding would be regarded as "instrumental". However, they were unable to understand the relationships and concepts involved. Further, it would appear that they did not check their work, thus making some avoidable errors.

Question 4

For the quadratic inequality, the responses from learner S and T are used. Learner S responded as follows:

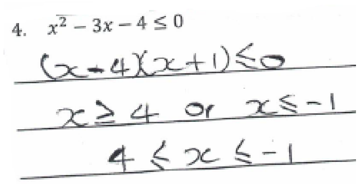


4. $x^2 - 3x - 4 \leq 0$

$(x + 1)(x - 4) \leq 0$

$\therefore x = -1$ or $x = 4$

Learner T's response:



4. $x^2 - 3x - 4 \leq 0$

$(x - 4)(x + 1) \leq 0$

$x \geq 4$ or $x \leq -1$

$4 \leq x \leq -1$

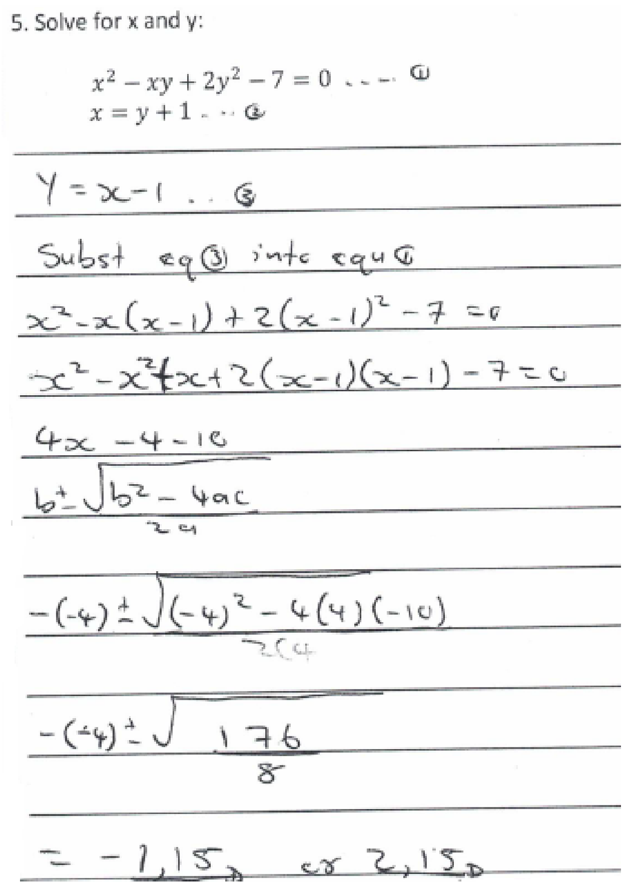
Learner S factorized the left hand side of the inequality and answered the question as an equation, probably not knowing what to do with the critical values obtained.

Learner T also factorized the expression correctly and knew that it was an inequality. However, it would appear that his answer was guesswork as he writes two separate inequalities and then attempts to combine the inequalities. Learner T's solution of $4 \leq x \leq -1$, while correct, does not follow from the previous step of $x \geq 4$ or $x \leq -1$ (which is incorrect). This shows a lack of understanding of the concepts involved when working with this inequality.

Question 5

For question 5 the responses of learners A and I are used.

Learner A:



5. Solve for x and y:

$x^2 - xy + 2y^2 - 7 = 0$ ... ①
$x = y + 1$ ... ②

$Y = x - 1$ .. ③

Subst eq③ into equ①

$x^2 - x(x-1) + 2(x-1)^2 - 7 = 0$

$x^2 - x^2 + x + 2(x-1)(x-1) - 7 = 0$

$4x - 4 - 10$

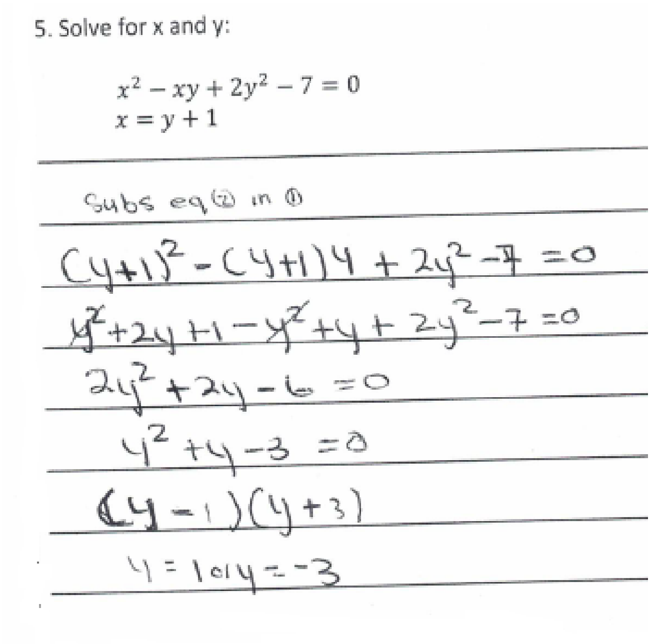
$\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$

$\frac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(-10)}}{2(4}$

$-(-4) \pm \sqrt{\frac{176}{8}}$

$= -1{,}15$ or $2{,}15$

Learner I:



5. Solve for x and y:

$x^2 - xy + 2y^2 - 7 = 0$
$x = y + 1$

Subs eq② in ①

$(y+1)^2 - (y+1)y + 2y^2 - 7 = 0$

$y^2 + 2y + 1 - y^2 + y + 2y^2 - 7 = 0$

$2y^2 + 2y - 6 = 0$

$y^2 + y - 3 = 0$

$(y-1)(y+3)$

$y = 1$ or $y = -3$

Learner A wrote $y = x - 1$ and was able to substitute for y in the quadratic equation. His simplification of the second step of his working resulted in an incorrect equation which he attempted to solve, not realizing that $4x - 4 - 10$ is a linear expression (not an equation) and the quadratic formula would not apply.

Learner I's second step was also incorrect simplifying $-(y+1)y$ as $-y^2 + y$ resulting in a quadratic equation which he attempted to factorize (even though it could not be factorized).

DISCUSSION

The test, that was set, was in line with question 1 of Mathematics grade 12 paper 1 and would appear to cognitively balanced with 75% of the marks allocated to knowledge and routine procedures. Learner performance was very average for the test. This would mean that those performing poorly are not likely to perform well in a similar exam question. Further, it would appear these poor performing learners would also have little or no chance of picking up enough marks in the rest of the examination to pass.

A scrutiny of some learners' answers for the first five questions of the test show that they were operating at a very instrumental level and applying procedures but were not able to make sense of the questions or their answers. In this regard, their understanding of the concepts involving the various components of quadratic equations was non-existent. Learners made very basic errors such as a failure to use the given form of the equation to obtain solutions and poor factorization. This would seem to be in line with Vaiyavutjamai, Ellerton and Clements's (2005) study on students' confusion when working with variables in quadratic equations. Other basic errors were incorrect simplification and not knowing what to do with the critical values of a quadratic inequality. Learners were also not able to work with a non-standard form of the quadratic equation. This is in keeping with Didiş, Baş & Erbaş's (2011) study which found that factorizing quadratic equations was challenging for students when presented in non-standard forms and structures.

Learners' errors may have been caused by them having a poor grasp of the basics or not having time to check their work. Further, it is rather unfortunate that these basic errors could be made by grade 12 learners, very late in the year. These types of errors should have been picked much earlier in grade 11 when learners are taught quadratic equations. These errors are also likely to surface in other areas of the curriculum where quadratic questions are applied (Govender, 2015).

RECOMMENDATIONS

There is no doubt that teachers should be aware of the way their learners work through various questions in mathematics and the types of errors they make. This will enable them to prepare their lessons with a view to addressing these shortcomings by their learners.

While this study focused on grade 12 learners, it is important for high schools to have a comprehensive plan for the teaching and learning from grade 8 onwards. The following recommendations are made, in this regard:

- Schools should **prioritise** mathematics teaching **in grades 8 and 9**. For example, at the beginning of grade 8, there should be a concerted effort to revise **key concepts** from primary school mathematics. Grade 10 teachers should **build** on what children have learnt from grades 8 and 9. In this regard, teachers should focus on **key content** and **skills** which learners need for the FET phase
- Many of the **errors** made by grade 12 learners appear to have their **origins** in **grades 8 and/or 9**. Teachers in these grades should know the **strengths** and **weaknesses** of their learners with a view to **building** on their strengths and **addressing** their weaknesses. This calls for **detailed** preparation on the part of teachers
- The author has also observed that grades 8 and 9 tend to be **neglected** at some schools. When grade 12 examinations start, there is a tendency for grade 8 & 9 learners to **stay away** because their examinations are a week or two away. Schools should use this time to have **special mathematics revision** classes for these learners as staying at home is likely to impact negatively on their performance
- When learners choose Mathematics as a subject in grade 10, it should be as a result of **good grade 9** results in Mathematics. Learners who **fail** grade 9 Mathematics are **unlikely** to succeed in grade 10 Mathematics

- It is important for mathematics teachers to discuss learner strengths and weaknesses in various topics at their **subject meetings** so that proper decisions regarding **learner support** are made
- Mathematics is a **high priority subject**. It should not be left only to the teacher to ensure that learners **develop** in the subject. **Parents**, irrespective of their levels of education, can also **support** this development. In these days of social and other media, teachers should **communicate** regularly with parents about their **children's progress** and what parents could **do** to support their children (Govender, 2007)
- While it is important for teachers to **cover** the prescribed content for mathematics, they should be mindful that the mathematics curriculum is **spiral** in nature, that is, learners see the same or similar topics during their school careers, but with each encounter increasing in **complexity** and **reinforcing** previous learning (Bruner, 1960). This means that there should always be revision of previous content covered in the same or previous years. This will ensure that by the time learners get to grade 12, they should be **fully prepared** for the last part of their school mathematics

CONCLUSION

This study has shown that some learners may sit in grade 12, having very little understanding of the basics required of them to work through questions in quadratic equations and inequalities, and possibly other topics. It is more than likely that these learners would have very little chance of passing Mathematics.

The recommendations, in the previous section of this paper, may offer solutions to schools where there is a high failure rate in Mathematics. Mathematics subject advisors and curriculum specialists should examine these recommendations carefully and include these as part of an overall strategy to improve learner performance in all schools.

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PROFILING MATHEMATICS OLYMPIAD LEARNERS

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Problem solving is one of the cognitive levels of assessment for Mathematics in the National Curriculum Statement (Grade R – 12) (commonly known as CAPS) in South Africa. This means that teachers should incorporate problem solving activities in their mathematics classes. For some learners this problem solving is not sufficient and they seek challenges outside the school. These learners are encouraged by their teachers and parents to take part in Mathematics competitions. This paper profiles senior high schools learners who have had some measure of success in these competitions. In this regard, 30 high school learners, from different schools in one Eastern Cape town, were selected for the South African Mathematics Team Competition in 2018. The writer facilitated three training sessions with these learners in preparation for the competition. At the conclusion of the competition, the learners were surveyed via a questionnaire. The writer uses the training sessions and the results of the survey to come up with a profile of these learners. This is done with the intention of making it easier for schools to identify and nurture such learners and may encourage more schools to provide mathematics enrichment opportunities for these learners.

INTRODUCTION AND BACKGROUND

Mathematics is a key subject in South Africa and other countries. One of the aims of the National Curriculum Statement Grades R-12, in South Africa, is to produce learners who are able *to "identify and solve problems and make decisions using critical and creative thinking*" (DBE, 2011). The term "problem solving" refers to mathematics tasks that have the potential to provide intellectual challenges for enhancing children's mathematical understanding and development (NCTM, 2010).

In the South Africa, four cognitive levels of assessment are prescribed. These are *Knowledge* (level 1), *Routine Procedures* (level 2), *Complex Procedures* (level 3) and *Problem Solving* (level 4) (DBE, 2011: 53). This means that learners should be exposed to problem solving activities from an early age.

Learners are also given exposure to problem solving when participating in Mathematics Olympiads and competitions. These Olympiads and competitions form an important part of the enrichment programme for mathematics learners in South African schools. The South African Mathematics Foundation (SAMF), in its website, discusses the importance of Mathematics competitions and the resultant enthusiasm and curiosity generated for Mathematics as a school subject. It also states that *"Mathematics is about thinking and the discovery, and validation, of problem solving methods"*. In this regard learners participating in Mathematics competitions stand to gain in the following ways:

- Participants in Mathematics competitions will be challenged by the problems and this will help improve their problem solving skills
- Problem solving skills can be further improved by carefully working through the solutions of the competitions
- Alternative and innovative solutions are given and the problems could be used in classroom discussions on problem solving
- There is a need for creative problem solving skills in today's technically oriented market place and expert problem solvers are needed. Practice in problem solving will help to train our future leaders of technological development (SAMF, 2019)

The South African Mathematics Olympiad (SAMO) consists of three rounds. Round 1 occurs in March, round 2 in May and round 3 in July each year. Learners require 50 % in round 1 to participate in round 2. Selection for round 3 is made on an invitation basis, usually the top 100 learners in round 2 of the SAMO (both junior and senior).

The Mathematics Olympiad season for school learners in South Africa ends with the ASSA South African Mathematics Team Competition (SAMTC) in September. Usually, top learners from rounds 1 and 2 are entered for the SAMTC. This competition, formerly known as the Interprovincial Mathematics Olympiad, has been a regular event since 1990. Teams from any province, district or region may participate in the SAMTC. Each region may enter a junior team (consisting of grade 8 & 9 learners) and a senior team (consisting of grade 10, 11 and 12 learners) and may also enter B, C and D teams at each level.

The first part of this competition consists of a one hour individual problem paper, consisting of 15 multiple choice questions. These questions are set similarly to those which appear in SAMO rounds 1 and 2. The individual papers are marked locally and the marks entered online on the SAMF website.

NB: SAMO (South African Mathematics Olympiad); SAMTC (South African Mathematics Team Competition)

After a break for refreshments, learners work in teams of 10 to answer the second part of the competition. The second part consists of difficult questions and team strategy plays an important role as each team only submits one set of answers. This team part of the paper is marked centrally by SAMF (SAMF, 2019a)

PROBLEM STATEMENT

Learners who participate in the SAMTC are usually those who have been successful in round 1 and/or round 2 of SAMO. These learners are usually good at problem solving and their levels of mathematical development are probably high.

It would be of interest to all those involved in mathematics and mathematics education to get a sense of who these learners are and how they were able to reach these high levels of mathematical development. This interest is the focus of this paper which profiles learners, from an urban region of the Eastern Cape, South Africa, who participated in the SAMTC in 2018.

LITERATURE REVIEW

The basis for most mathematics problem solving research for the secondary school is Polya (1973). Polya proposed four phases in problem-solving: ***understanding the problem, devising a plan, carrying out the plan*** **and** ***looking back*** .In his four-phase plan and other works, he suggests a list of heuristics, which include the following:

> *Drawing a figure; looking at simpler cases to search for a pattern; modifying the problem (replacing given conditions by equivalent ones; recombining the elements of the problem in different ways); exploiting related problems (simpler problems, auxiliary problems, analogous problems); working backwards; arguing by contradiction or contrapositive; decomposing and recombining; generalizing; specializing.*

Heuristics do not have any particular role out of the problem solving context but can be quite powerful when incorporated into situations of doing mathematics (Wilson, Fernandez and Hadaway, 2006). Garofola and Lester (1985) have suggested that students are largely unaware of the processes involved in problem solving and that addressing this issue within classrooms and other problem solving forums should be a priority. To become a good problem solver in mathematics, students should have a good mathematical knowledge base and be able to organize that knowledge. This appears to support Kantowski (1977) who states that students with a good knowledge base were most able to use the heuristics in geometry.

Schoenfeld and Herrmann (1982) found that those working with problems for the first time looked at the surface features of the problems while those who were experts categorized problems on the basis of the mathematical principles involved. This is in line with an earlier finding by Silver (1979) who reported that successful problem solvers were more likely to categories mathematics problems on the basis of their similarities in mathematical structure.

Murray, Olivier and Human (1998) describe problem solving as a *vehicle* for learning mathematics. They state that it is necessary to distinguish between learning *to* solve problems and learning *through* solving problems. Learning *to* solve problems would require learners to know the mathematics content and some mathematical techniques to solve problems. These points are also echoed by the Ontario Ministry of Education, which states that "*By learning to solve problems and by learning through problem solving, students are given numerous opportunities to connect mathematical ideas and to develop conceptual* *understanding"* (Ontario Ministry of Education, 2005:11). Thus, it would be fair to say that problem solving is regarded as a foundational building block for learning mathematics.

Learning how to solve problems can be established as a long term teaching and learning process and encompasses four phases (Bruder and Collet, 2011) , cited in Liljedahl, Santos-Trigo, Malaspina & Bruder (2016). These phases are:

- Intuitive familiarisation with heuristic methods and techniques
- Making aware of special heurism by means of prominent examples
- Short conscious practice phase to use the newly acquired heurism with differentiated task difficulties
- Expanding the context of the strategies applied

These phases, which build on from the research of Polya and others, offer teachers a structured way to develop problem solving competences among learners in their classrooms.

Lester (2013) lists seven key principles which should stimulate discussion and underpin problem solving research. These are:

> *"prolonged engagement; task variety; complexity; systematic organization; multiple roles for the teacher; group interaction and assessment".*

All of these principles are important and for purposes of this study, five of these principles are discussed here.

Prolonged engagement

In order for students to improve their ability to solve mathematics problems, they must engage in work on problematic tasks on a regular basis, over a prolonged period of time.

Task variety

Students will improve as problem solvers only if they are given opportunities to solve a variety of types of problematic tasks

Complexity

The interaction, between mathematical concepts and the processes used to solve problems involving those concepts, is dynamic and involves heuristics, skills, control processes, and awareness of one's own thinking and these take place concurrently with the development of an understanding of mathematical concepts.

Systematic organization

Problem solving instruction, in particular metacognition, is likely to be most effective when it is given in a systematically organized manner under the direction of the teacher.

Multiple roles for the teacher

Problem solving instruction that emphasizes the development of metacognitive skills should involve the teacher in three different, but related, roles: an external monitor, a facilitator of students' metacognitive awareness and as a model of a metacognitive adept problem solver.

These principles appear to consolidate what has been discussed in this literature review, which may be summarized as follows:

- Polya's four phase plan and his list of problem solving strategies are very important when introducing problem solving in classrooms
- Students should be aware of the processes involved in problem solving through engagement with suitable problems; they need to have a good mathematical knowledge base and should know how to organise that knowledge when solving problems
- Problem solving is a developmental process. Students who are novices at problem solving are not able to engage deeper into the problems; those who have the necessary experience are able to understand the mathematical principles underlying the problems and select appropriate strategies to solve the problems
- Problem solving may be used a vehicle for learning mathematics and also serves as a foundational building block for the subject
- Learning how to solve problems can be established as a long term teaching and learning process
- There are key principles which stimulate discussion and underpin research in problem solving. These cover a number of issues, including, time spent on problem solving, working with a variety of tasks which increase in complexity, being able to organise information systematically and the roles of the teacher.

RESEARCH QUESTION

In the light of the introduction and background, as well as the literature review for this study, the following research question was posed:
What are the attributes of learners who are successful in Mathematics Olympiads?

The following subsidiary questions were set, within the context of the research question, for this study:

- What are the views of Mathematics Olympiad learners on the Mathematics and how would they describe their performance in the subject?
- Why do these learners participate in Mathematics Olympiads and for how long have they been participating?
- What do these learners say about training received for the SAMTC?
- What were the comments of learners on the questions set in the individual round of the SAMTC and how do these compare with their actual performance

- How do learners feel on being selected to participate in the SAMTC?

SAMPLE

The sample for this study consisted of 30 grade 10 to 12 learners who were selected to participate in the South African Mathematics Team Competition in 2018. They were from schools from an urban area of the Eastern Cape and participated voluntarily in this study.

CONCEPTUAL FRAMEWORK

This study has both an ontological and epistemological basis (Cohen & Manion, 1985). The writer knew that the learners, who formed the sample for this study, had done well in Mathematics Olympiads and competitions during the course of 2018 and were selected to participate in the SAMTC on the basis of these results. Thus, these learners had some experience in Mathematics Olympiads and probably had views on a number of aspects with regard to Mathematics and Mathematics competitions. This forms the ontological basis for this study.

The process of finding out what these experiences and views are is in line with the epistemological basis of this study which is to come up with a profile of learners who were successful in Mathematics Olympiads.

In South Africa, Mathematics Olympiads are regarded as an "add-on" for learners and teachers. Learners tend to work through Olympiad- type questions in their own time and their teachers may or may not give them support. However, these Olympiads tend to enrich the learners' mathematical knowledge. Schools may use Mathematics Olympiads and competitions to promote excellence in mathematics learning, develop and enhance self confidence in learners and also nurture creativity amongst the learners.

In this regard, the Enrichment Triad Model (ETM) may be relevant for this study (Renzulli, 1977). This model consists of three different kinds of interrelated forms of enrichment activities that are integrated as a complement to the regular curriculum. Of these, type I enrichment is relevant to this study. It consists of general exploratory experiences that are designed to expose students to topics and areas of study not ordinarily covered in the regular curriculum. In this study, the enrichment activities involved learners using their problem solving knowledge to work through Mathematics Olympiad questions. Thus, Polya's four step approach to problem solving is also relevant. Thus an appropriate framework for this study is ***problem solving within an enrichment context.***

As stated earlier, this study was geared to find out the views and experiences of the learners with regard to Mathematics and Mathematics Olympiads and involves both a description and an interpretation of the data collected, thus locating the study within the interpretive research paradigm (Bassey, 1999).

RESEARCH METHODOLOGY

This research involved the collection of both qualitative and quantitative data. The views and experiences of the learners were the objects of this research thus, ensuring that the research was mainly qualitative in nature (Hatch, 2002). Quantitative data (the marks obtained in the individual part of the SAMTC) was also collected in this study.

DATA SOURCES

Data for this study were collected from a number of sources. These included:

- Learner selection for the SAMTC
- The training programme in preparation for the SAMTC
- Questionnaires administered to the learners after they had finished competing in the SAMTC. The questions sought information from learners on a variety of matters:
  their views on Mathematics as a school subject; their performance in school mathematics; why they participated in Mathematics Olympiads; their experience in participation in Mathematics Olympiads; training received and what was learnt from the training; their selection to participate in the SAMTC; rating of questions in the paper; how they worked on team questions; participation in the SAMTC in 2019;; career choices (grade 12 learners)
- The marks obtained for the individual questions of the SAMTC.

RESULTS

Learner selection for the SAMTC

The writer used the results of SAMO round 2, in the main, to select three teams (A, B and C) in July 2018 to participate in the SAMTC. Each team consisted of 10 members. To ensure that each team had the required number of members, the writer also used performance in SAMO round 1 as an additional criterion.

Training programme

The training programme consisted of three two hour sessions with the members of the three teams. The programme consisted of the following activities:

Session 1: Presentation to team members on the SAMTC. All team members were given a background to the SAMTC, the region's involvement in the SAMTC, past performance the teams in the SAMTC, format of the competition and discussion of individual questions from past papers

Session 2: Discussion of individual questions from past papers; learners work on their own and in pairs

Session 3: Discussion of team questions from past papers; discussion of team strategy; learners work in pairs

During the training programme as described above, Polya's and Bruder and Collet's problem solving steps , as discussed earlier in the literature review, were used when the various problems were discussed. Learners were made familiar with the various methods and techniques involved in problem solving. Examples from past year papers were used to provide learners with practice in these methods and techniques. Two examples are shown here:

Example 1 (individual question)

The last digit of $(7^8)^{2016} + (9^{10})^{2016}$ is [6 marks]

(A) 0	(B) 2	(C) 4	(D) 6	(E) 8

Step 1 Understand the problem: We note that the problem consists of working out a sum involving exponents having 7 and 9 as bases to establish whether there are any patterns

Step 2 Devise a plan: We will simplify the problem by working through powers of 7 and 9 separately

Step 3 Carry out the plan:

$7^1 = 7; 7^2 = 49; 7^3 = 343; 7^4 = 2401$

Pattern of last digits is 7; 9; 3; 1; 7; 9; 3; 1

$9^1 = 9; 9^2 = 81; 9^3 = 729; 9^4 = 6561$

Pattern of last digits is: 9; 1; 9; 1

Last digit of 7^{8x2016} is 1 (power is a multiple of 4)

Last digit of $9^{10x2016}$ is 1 (power is an even number)

Step 4: Looking back:

Check the patterns involved and the solution obtained. It is one of the choices given. Thus, the last digit of $7^{8x2016} + 9^{10x2016}$ is 1 + 1 = 2 (the answer is B)

Example 2 (individual question)

If a, b and c are numbers that belong to $\{0; 1; 2; ...\}$ and are such that $a(a+b)(a+b+c) = 11$, find c.

[6 marks]

(A) 0	(B) 1	(C) 10	(D) 11	(E) Cannot be found

Step 1 Understand the problem: We note that the problem consists of three factors all of which when multiplied out will give an answer of 11

Step 2 Devise a plan: We note that since none of a, b and c are negative, $a \leq a+b \leq a+b+c$. We also know that $1 \times 1 \times 11 = 11$ (no other integral factors are possible)

Step 3 Carry out the plan:

Thus, we may say that a = 1; a + b = 1 and a + b + c = 11. So b = 0 and c = 10

Step 4 Looking back:

We note that with a = 1, b = 0 and c = 10 we have

(1)(1+0)(1+10+0) = (1)(1)(11) = 11. Thus, our answer is C

Data from questionnaires

Learners were given codes: **S101; S102; …; S110; S201; S202; …;S210; S301; S302;…; S310**. The questionnaires were analyzed with a view to identifying emergent trends and patterns of coherence.

Mathematics as a school subject

Most learners were very positive about school mathematics. They used words such as *"problem solving subject"; "fun and enjoyable"; "critical thinking"; "rewarding"; "very engaging"; "an interesting and complex subject"; "quite easy"; "builds up to all the other science subjects".*

The following learners gave different views:

> **S102** stated that it is *"normally a bit boring when you do the same thing over and over again"* ; **S306** remarked that it is *"A daily 50 minutes of mental torture that is necessary for my future"*

Performance in school mathematics

All learners indicated that they did very well in school mathematics with most indicating that they regularly obtained 90% and above in the subject. The views of three learners, who did not give any percentage, are captured here:

S205 stated that *"it is not the level I want to be but for the time being I am satisfied with my performance"*; **S301**reported *"I am in the second top set which surprised me; I did not start off well but did well in the end which is why I am in the high set"*. **S309** stated that *"I am above average but below excellent; it has slipped because of my music responsibilities (violin).*"

Participation in Mathematical Olympiads

All the learners surveyed indicated they loved participating in Mathematics Olympiads. They used words such as *"love the challenge"; "interesting"; "learn new things"; "helps to sharpen my school mathematics"; "improve my problem solving skills"* as reasons for participation in Olympiads. Only one learner **(S205)** indicated that he was influenced by his teacher to participate in Mathematics Olympiads. Of interest in this study would be the experience which these learners had in Mathematics Olympiad participation. This information appears in Table 1.

Table 1 Years of participation in Mathematics Olympiads

Number of years	Number of learners
1	2
2	1
3	2
4	6
5	10
6	3
7	5
8	1

25 learners (83,3%) have been participating in Mathematics Olympiads for four years or more. This would indicate that the majority of the learners selected for the SAMTC were very experienced in Mathematics Olympiads. Interestingly, one of the first time participants was in grade 12. It would be fair to say that the majority of learners have had a **prolonged engagement** with problem solving type questions which form part of Mathematics Olympiads (Lester, 2013).

Mathematics Olympiad Training

In arranging training for the SAMTC, the facilitator ensured that learners had to be exposed to a variety of problems with different levels of **complexity** (Lester, 2013). The facilitator, who was experienced in preparing teams for the SAMTC, did this in a structured manner in the sessions available for the training. Only three of the learners had not participated in the training for the SAMTC training. These learners were busy with their grade 12 trial examinations and the training materials were sent to these learners. A further three learners were present for one or two sessions. The rest attended all three SAMTC training sessions. 12 learners indicated that they had attended the SIYANQOBA training sessions earlier in the year. Some schools also had training for their learners. Six learners attended the school training.

All learners found the training for the Mathematics Olympiads very beneficial, using words such as "*interesting"; "informative"; "helpful"; "good warm up"; "knowledgeable"; "challenging"; "good to work with others"* to describe the training received for the SAMTC.

Selection for the SAMTC

All learners were extremely proud and found it to be an honour and a major achievement to be selected for such an important completion. They wanted to be role models for other learners at their schools and hoped to do well in the competition. Their parents, teachers and friends congratulated them on this fine achievement and gave them words of support and encouragement.

Standard of questions in the individual part of the paper

Learners reported that the first few questions were easy but as they moved on, the questions became more difficult, with the last five being extremely difficult. Some learners indicated that they did not know how to start with some of the questions and struggled to finish within the allotted time.

Working on the team questions

Each team used a similar strategy when doing the team section of the SAMTC. The team captain and his assistant allocated questions according to the "perceived" strengths of each member of the team. They worked in pairs and each pair was given two questions to do in the first 30 minutes.

The full teams sat together for the second 30 minutes, with the captain as the chairperson. Each pair then discussed their responses to their questions with the whole team. If pairs had difficulty with any question, then other members of the team assisted them during this discussion. In the end, using consensus, the final list of answers was compiled.

Learner **S305** remarked that it was a "*unique experience to do mathematics as a group effort; you have to listen to other people's opinions and explain your own working"* . Another learner, **S308** stated that "*Team work always conquers all; whether the answer is correct or not"*.

Participation in the SAMTC in 2019

18 participants were in grade 10 or 11. These learners indicated that they would definitely want to participate in the SAMTC in 2019. They were grateful for the

experience gained in the competition as they enjoyed working through challenging questions with like-minded colleagues from other schools; this will contribute to their further development in mathematics. Grade 11s also indicated that 2019 would be their last year in school and welcomed the opportunity of a final participation in the competition where they could play a mentoring or support role to junior members of their teams.

Tertiary education (grade 12 learners)

12 participants were in grade 12. For these learners the SAMTC was the last opportunity to be involved in a Mathematics competition aimed at school learners. Of interest to this study would be the types of tertiary programmes these learners would choose and whether their participation in Mathematics Olympiads would influence this choice. In this regard, the World Economic Forum in its Future of Jobs Report (WEF, 2016) has "complex problem solving" as number one in its top 10 skills required for jobs in both 2015 and 2020. It was with key skill in mind that these learners were asked about their tertiary education plans (and career choices) and how their participation in Mathematics competitions and exposure to high levels of problem solving has prepared them for tertiary education. These are shown in Table 2:

Table 2 Career choices: Grade 12 participants

Learner code	**Career choice**	**Reasons**
S101	Actuarial Science	Looking for solutions when assessing risk in insurance and finance industries
S103	Mathematics & Science teaching	Problem solving approaches will be included in my teaching
S104	Arts	Doing creative and look at different and innovative solutions
S108	Medicine	Problem solving approaches helps when diagnosing illnesses
S109	Computer Science	Problem solving features strongly in the computer sciences especially in programming
S203	Psychology or Statistics	Psychology: Using problem solving approaches to understand how people think Statistics: Work and analyze numbers

S204	Forestry	Involves lots of problem solving and innovative thinking
S205	Engineering (Mechatronics)	Help in identifying problems/errors in code and fix these problems/errors
S206	Medicine or Health Sciences	Help me solve problems in the field
S208	Medicine	Help in finding solutions, e.g.) positioning for x-rays and treating of patients
S302	Medicine	Diagnose illnesses and find cures to diseases
S304	Engineering (Mechatronics)	Solve tough problems

Table 2 shows a wide variety of career choices by these grade 12 learners. All agreed that their participation in Mathematics Olympiads/competitions and their exposure to high levels of problem solving would play a significant role in their future careers. This falls in line with the top work skill of "creative problem solving" as indicated in the World Economic Forum report.

ADDITIONAL COMMENTS

Learners were given the opportunity of giving additional comments, not necessarily covered in other questions. These comments are listed below:

> *"I love this competition"; "SAMTC is different from others; prepares you for many aspects of life including the working world where people have to come together, collaborate and work as a team to solve problems and find solutions"; "It was very tough and blew me away; more complicating when compared to 2017; I wish I could be here next year"; "I would just like to say that overall I enjoyed my experience during these workshops and it improved my maths skills"; "I really enjoyed it; at first I was rather skeptical but in the end I enjoyed the challenge and meeting new people"; " Schools should do training during the week; love to have an in school competition"; "A fun experience".*

There is no doubt that the SAMTC left an indelible mark on the learners who participated in the competition.

Learner performance in the individual round

Learner performance, in questions of the individual round of the SAMTC, is shown in Table 3.

Table 3 Learner performance in the individual questions

Question	Number correct
1	16
2	6
3	8
4	20
5	20
6	5
7	5
8	5
9	1
10	28
11	11
12	16
13	5
14	11
15	10

It would appear that learners struggled with questions 2, 3, 6, 7, 8, 9 and 13 while performance in questions 1, 4, 5, 10 and 12 being quite reasonable, especially question 10. All questions are shown later in this paper in annexure A.

FINDINGS

The findings of this research are written with the research question and subsidiary questions in mind. It is now opportune to recall the research question:

What are the attributes of learners who are successful in Mathematics Olympiads?

It is now possible to answer this question, using the literature review and the data collected, including analyses thereof. The attributes of learners who tend to be successful in Mathematics Olympiads/competitions may be stated as follows:

- These learners regard Mathematics as an **important school subject** and have a great affinity of mathematics. However, some of these learners get **"bored"** with school mathematics and tend to seek **outside challenges** to help develop them further in mathematics.

- They do **exceptionally well** in school mathematics, usually getting **more than 90%** in school tests and examinations. Some learners get close to full marks in their school assessments. It would appear that school mathematics is **not a challenge** for these learners
- As indicated in bullets 1 and 2 above, these learners have reached a high level of school mathematics and do not find school mathematics challenging enough for them. They seek these challenges in **Mathematics Olympiads/competitions**. The majority of them have been participating in competitions for **four years or more**. This means that they have been involved in these competitions for **most** of their high school careers; some had their earliest taste of competitions in grade 3. These learners are likely to look deeper into the problems based on the mathematical principles involved (Schoenfeld and Herrmann ,1982), and applying the principles of Polya and Bruder and Collet's problem solving steps has probably become second nature to them.
- Despite these learners having a high level of mathematical development, they are on the look out to **further** this development. In this regard, most of them have been involved in some **Mathematics Olympiad training** during the year. These include the **SIYANQOBA** training programme held in earlier in the year as well as training for the SAMTC. Learners from one of the schools are also used as **tutors** in the school's enrichment programme.
- These learners represent some of the **best** Mathematics Olympiad learners in the region and they were **appreciative** that their efforts during the year did not go unnoticed. All of them were grateful for being selected for the SAMTC. Their teachers, parents and others **congratulated** them on this great achievement. Despite being **competitive** and striving to be the best, the learners welcomed the opportunity of working collaboratively with other **"like-minded"** pupils in their respective teams. They were able to learn more from their **peers**, especially how they approached their quite difficult team questions
- Their performance in the individual paper appeared to be in line with their comments on the questions. Despite the **difficulty** of a number questions (in their opinions), they **relished** the challenge posed by the paper. In fact, grade 10 and 11 learners in the different teams indicated they would welcome the opportunity to **participate** in the SAMTC in **2019**.
- For grade 12 learners, the SAMTC represented the last opportunity to participate in a Mathematics competition with some saying they will miss the competition. These learners believe that their use of various **heuristics or strategies** in working through the mathematics problems in these competitions is likely to assist them in their future careers. These include some very **high** level careers.

CONCLUSION

When the National Examination grade 12 results are announced in South Africa each year, there is always a close scrutiny of the Mathematics results by various education experts. This scrutiny often results in negative comments, despite progress made.

This study, despite its small sample of 30, has shown that there are learners in our schools who love Mathematics and do well in the subject. Many of these learners get the opportunity of participating in Mathematics competitions. Through training and interaction with other like-minded learners, they become more confident in their ability to solve mathematics problems in the competitions. This, in turn, leads them to doing better in school mathematics.

While passing mathematics is important, there should be an emphasis on improving the quality of these passes. If teachers promote problem solving in their classrooms and enter their learners in Mathematics competitions (and expose them to relevant training), there is a strong possibility of the quality of mathematics passes improving. This study has shown that learners who do well in Mathematics and participate in Mathematics competitions are likely to choose high level careers when they leave school.

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Annexure A South African Mathematics Team Competition:

Individual Round: Seniors

1. If 6 cats can eat 6 mice in 6 minutes, how many cats would it take to eat 100 mice in 100 minutes? [6 marks]

(A) 100	(B) 36	(C) 6	(D) $\frac{100}{6}$	(E) 600

2. A farm has a certain number of black cows and certain number of brown cows. Suppose all the cows of the same colour give a constant amount of milk everyday. Four black cows and three brown cows give as much milk in five days as three black cows and five brown cows give in four days. What is the ratio of the amount of milk given by a black cow to that of a brown cow? [6 marks]

(A) $5:8$	(B) $7:9$	(C) $2:1$	(D) $8:5$	(E) $5:4$

3. A race car is going around a track that is a square, one kilometre on a side. If it goes at an average speed of 60 km/h on the first two sides and 90 km/h on the third side, how fast must it go (in km/h) on the fourth side to average 60 km/h all the way around? [6 marks]

(A) 30	(B) 60	(C) 45	(D) 50	(E) 40

4. At midnight, the hour and minute hands of a clock overlap. Through what angle (in degrees) must the hour hand of a clock turn between this overlap and the next overlap? [6 marks]

(A) 30	(B) $\frac{360}{11}$	(C) $\frac{330}{12}$	(D) 60	(E) $\frac{330}{11}$

5. In a convex quadrilateral $ABCD$, $\angle DAB = \angle ABC = \theta$. The angle bisectors of $\angle CDA$ and $\angle BCD$ intersect at a point E inside the quadrilateral. Then $\angle CED$ must be equal to [6 marks]

(A) θ	(B) 2θ	(C) $\frac{1}{2}\theta$	(D) 3θ	(E) 4θ

6. If $\sqrt{x+a}-\sqrt{x-a}=\sqrt{2}$, then x is equal to [6 marks]

(A) a	(B) $\frac{1}{2}(1+a^2)$	(C) $\frac{1}{2}(1-a^2)$	(D) $\frac{1}{2}(1+a)$	(E) $\frac{1}{2}(1-a)$

7. A positive integer d divides both n^2+1 and $(n+1)^2+1$ for some integer n. How many possible values of d are there? [6 marks]

(A) 0	(B) 1	(C) 2	(D) 3	(E) Infinitely many

8. In a convex N-gon (polygon with N sides, $N \geq 3$), what is the maximum number of exterior angles that can be obtuse? [6 marks]

(A) $N-2$	(B) $N-4$	(C) $\frac{N}{2}$	(D) 3	(E) 4

9. How many multiples of 2018 contain only the digits 7 and 0? [6 marks]

(A) 0	(B) 1	(C) 1009	(D) Infinitely many	(E) 18

10. Themba has N coins. He distributed them in 20 initially empty lockers in such a way that no locker is empty now and each locker has a different number of coins. What is the minimum value of N? [6 marks]

(A) 210	(B) 150	(C) 55	(D) 231	(E) 275

11. Let a, b and c be positive real numbers such that $abc=1$. Let $d=a^{(b+c)}b^{(c+a)}c^{(a+b)}$. What is the maximum possible value of d? [8 marks]

(A) $\frac{1}{3}$	(B) 1	(C) $\sqrt{3}$	(D) $\sqrt{2}$	(E) 3

12. The increasing sequence $3, 15, 24, 48, \ldots\ldots$, consists of those positive multiples of 3 that are one less than a perfect square. What is the remainder when the 2018 th term of the sequence is divided by 9? [8 marks]

(A) 0	(B) 5	(C) 7	(D) 6	(E) 4

13. Consider a triangle ABC. Points M and N lie on the line AB while M' and N' are on AC, such that MM' and NN' are parallel to BC. These parallel lines divide the triangle in 3 equal areas. Then the ratio $AM : AN$ is [8 marks]

(A) $1:2$	(B) $1:\sqrt{2}$	(C) $1:3$	(D) $1:\sqrt{3}$	(E) $\sqrt{2}:\sqrt{3}$

14. This year there are two months in which the 13th day is a Friday (April and July). When is the next year in which there is just one month in which the 13th day is a Friday? [8 marks]

(A) 2019	(B) 2020	(C) 2021	(D) 2022	(E) 2023

15. Let the diagonals of a trapezium $ABCD$, (AB parallel to DC and DC is the base), intersect at O. Given that one of the base angles is double the other. The parallel to AB through O meets AD and BC in M and N respectively. Which of the following is true?

[8 marks]

(A) $AM = 2NC$	(B) $AO = OD$	(C) $OM = ON$	(D) $AD = 2BC$	(E) $OD = 2OA$

MATHEMATICS TEACHING ENVIRONMENTS AS EXPERIENCED BY MATHEMATICS TEACHERS IN THABA NCHU SECONDARY SCHOOLS

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Teaching mathematics in a township school can be challenging in many ways. Mathematics teachers in Thaba Nchu secondary schools implement a unique way of teaching to cope with the challenges and obstacles they experience in their teaching environments. Five mathematics teachers from three secondary schools in Thaba Nchu participated in this study. The focus of the study was to give the participants an opportunity to share their feelings and experiences about teaching in their unique environments. Semi-structured interviews were used to capture the participants' stories. The analysis and interpretation of the interviews resulted in the identification of challenges and positive aspects that each participant experiences in his/her teaching environment.

INTRODUCTION

A teaching environment can be thought of as any environment in which a teacher finds him/herself while in the act of teaching. The environment can therefore range from a high-technology teaching laboratory to the area around a dusty chair under a tree. Most teachers are highly adaptable when having to adjust to their teaching environment, as the endless list of factors that influence and change the teaching environment are omnipresent, and have to be handled on a daily basis.

Factors like teachers' values, attitudes and beliefs, and stress caused by certain aspects of the teaching environment, influence teaching. Some of the most general causes of teacher stress include learners who are not interested in the subject, learners causing disciplinary problems, general time constraints, a multitude of tasks at hand, adjusting to changes in the teaching environment, the prescribed curriculum, teacher-evaluation processes, interaction with demanding colleagues, low self-respect and self-confidence, and substandard working environments (Mujtaba & Teiss, 2013).

Learners greatly influence the teaching environment. Other than the learners' ability to master the content, low learner self-efficacy is a factor that the teacher has to deal with in the classroom (Winheller, Hattie & Brown, 2013) and which effects learners' academic performance directly (Winheller et al., 2013). Unqualified or under-qualified teachers are a challenge in many schools (Consortium for Policy Research in Education: Policy Bulletin, 2001). This leads to ineffective teaching and thus to ineffective learning. Many other factors like outdated textbooks, old technology such as photocopy machines and computers that don't function as needed, and ill-mannered learners, also have a considerable influence on a teacher and his/her teaching.

All of the mentioned factors form part of the teaching environment experienced by teachers on a daily basis. In this study, the researcher aimed to give the participants a platform where they could share the challenges and positive aspects of their teaching environments and how they experience it. In literature, the focus of most research in this field is on the learner, the learning environment, and how to improve it (Ediger, 2012; Winheller et al., 2013; Shamaki, 2015; Sharma, 2015). Teachers are seldom given the opportunity to reflect on the challenges they encounter and on suggestions about how to overcome these challenges.

PURPOSE AND SIGNIFICANCE OF THE STUDY

The purpose of the study was to understand how mathematics teachers in Thaba Nchu secondary schools experience the challenges as well as the positive aspects of their teaching environments. No literature regarding the teaching environments of mathematics teachers from Thaba Nchu secondary schools is available. This observation makes the study significant and therefore adds to the knowledge regarding teaching in the rural South African secondary school setting.

The purpose and significance of the study as outlined here, directly leads to the research question that is answered in this paper.

RESEARCH QUESTION

How do practicing mathematics teachers in Thaba Nchu secondary schools experience their teaching environments?

THEORETICAL FRAMEWORK

Merriam (2009) refers to qualitative researchers as researchers who want to understand how their participants make sense of their experiences in social and cultural contexts. This statement serves as support for this study’s research question and thus the research aim. The theoretical framework of this study is formed by autoethnography and structuralism. These are explained in the following paragraphs.

Auto ethnography

In many literature sources, auto ethnography is referred to as a qualitative research method; however, in this study, it forms part of the theoretical framework as well. The reason being that the researcher used her own experiences and perspectives of teaching to understand the social and cultural aspects of the teaching environments of the participants. Adams, Jones & Ellis (2014) did comprehensive research on auto ethnography and concluded that auto ethnography could be used as part of the theoretical framework of a study.

Auto ethnography is a research proposition used to investigate (*-graphy*) the researcher's personal involvement (*auto-*) to make meaning of cultural experiences (*-ethno-*) (Ellis, Adams & Bochner, 2011; Ellis, 2004; Holman Jones, 2005). Auto ethnography thus entails writing about a culture which the author or researcher is part of (Munro, 2011). According to Ellis *et al.* (2011) autoethnographers acknowledge the many ways that a researcher's personal involvement and experience can have an impact on the research process. Sparkes (2000) defines auto ethnography as research that uses the researcher's personal experience to understand the social context. According to Laslett (1999) auto ethnography is a mixture of the researcher's personal experiences and what the community gives, to make a new contribution to social sciences.

Structuralism

Structuralism lets the researcher understand parts of culture through their relationship with other parts to form a larger structure. The Shmoop Editorial team (2008) points out that structuralists aim to find and study the structures that are fundamental to all cultural experiences. Structuralists are searching for the understanding of the deep structure of cultural and social aspects (Shmoop Editorial Team, 2008; Barker, 2004). According to structuralists, Structuralism can be used in any situation, experience or aspect to learn about the deeper structure defining the obvious structure as we see it (Shmoop Editorial Team, 2008). Structuralism is, therefore, an understanding of the essential patterns that influence our behaviour, the way we speak and how we act.

Education consists of many aspects of which teaching environments are one. Teaching environments consist of many other sub-aspects (or structures) like teachers, teacher training, physical aspects of the environment, social aspects of the learners, the community, and affective aspects. Each one of these aspects cannot, on its own, provide a proper understanding of the teaching environment. By studying all the aspects and understanding the relationships between the aspects, an understanding of this study's system (teaching environments) and its underlying structures, is obtained.

LITERATURE REVIEW

The schools that formed part of the study are located in a rural township area. Therefore, it is necessary to review township and rural schools and the challenges associated with these schools.

Often the terms 'rural schools' and 'township schools' are used in similar and sometimes confusing ways. Township residential areas are defined by Mampane & Bouwer (2011) as areas originally formed to isolate certain races. These areas have low-cost housing with the main aim of accommodating black workers close to their places of employment. Schools in these areas are referred to as township schools. Another definition by Badenhorst & Badenhorst (2011) explain township schools as those in black communities, which originated during the apartheid era. This definition is supported by Ndimande (2012) who defines township schools as schools that exist in the urban areas previously isolated during the apartheid era.

People, in general, have an idea what a rural area is, but the official definition is not precise. Rural areas are more commonly referred to as areas that are characteristic of the countryside, rather than the town. In other words, rural areas are removed and often underdeveloped (Du Plessis, 2014). Pillay and Saloojee (2012) classify rural settings as areas with unqualified workers and unemployed people. Schools within these settings are referred to as rural schools. Monk (2007) logically summarizes the term rural as everything that is not urban or metropolitan. In general, both township and rural areas refer to areas that are characterised by low socio-economic circumstances, a low percentage of qualified residents and a low level of employment.

Challenges experienced in township or rural schools has become a popular research topic over the past few years. Challenges are categorised into physical challenges, socio-economic challenges, teacher-faced challenges and management challenges.

Physical challenges include aspects like the lack of running water (Lumadi, 2014; Moloi & Kamper, 2010; Mukeredzi, 2016), poor or insufficient toilet facilities (Lumadi, 2014; Mukeredzi, 2016), the unavailability of electricity (Mafora, 2013; Adedeji et al., 2011; Du Plessis, 2014), substandard facilities (Christie, 1998; Mukeredzi, 2016), the poor physical state of classrooms (Buckley, Schneider & Shang, 2004; Mafora, 2013), limited resources on all levels (Waller & Maxwell, 2011; Moloi et al., 2010; Mukeredzi, 2016), and difficulty reaching the schools (Mafora, 2013; Moloi et al, 2010; Du Plessis, 2014).

Socio-economic challenges include low parental involvement (Cotton & Wikelund, 1989; Singh, Mbokodi & Msila, 2004; Modisoatsile, 2012; Waller et al., 2017), the poor socio-economic state of the communities surrounding the schools (Mampane et al., 2011; Mukeredzi, 2016), government funding and no school fees (Adedeji et al., 2011; Du Plessis, 2014; Mampane et al., 2011), the reality of child-headed households (Pillay & Nesengani, 2006) and a high level of learner absenteeism (Du Plessis, 2014; Taylor & Yu, 2009).

Teacher attrition and teacher retention (Manual, 2003; Croasmun, Hampton & Herrmann, 1997; Ewing & Smith, 2003), insufficient teacher training (Mukeredzi, 2016; White, 2011; Richards, 2012), learners challenging discipline in class (Rossouw, 2003; Serame, Oosthuizen, Wolhuter & Zulu, 2013), unqualified and insufficient teachers (Lumadi, 2014; Badenhorst et al., 2011), overcrowded classrooms (Badenhorst et al., 2011; Lumadi, 2014) and high teacher absenteeism (Du Plessis, 2014; Morar, 2003) are a few of the challenges faced by teachers in township and rural schools.

Another category of challenges in township and rural schools is that of management. This category includes aspects like principals' management-styles (Mahlangu, 2014; Du Plessis, 2014) and the involvement of the School Governing Body (SGB) (Mahlangu, 2014). Being a teacher in a township or rural school requires teachers with nerves of steel. Despite all the challenges highlighted in literature, teachers are still expected to be productive and deliver high-quality education. Poor results demand explanations (Lumadi 2014:247). Very few teachers can keep delivering their best in such conditions.

METHODOLOGY

A qualitative research approach was followed to successfully answer the research question within the framework of auto ethnography and structuralism.

Paradigm, ontology and epistemology

According to Creswell (2009), individuals form subjective meanings of experiences and situations. Many of these meanings are formed as a result of interactions with other individuals or groups of people. The paradigm of this study is therefore social constructivism as it focuses on determining how people interpret and understand their experiences in their living environment, and how these experiences impact on the meanings that are socially constructed. The researcher's ontological stance in this study is relativism, as the data in this research is not fixed, but flexible. The participants' reality, in this case, is socially constructed. By reflecting on the interactions of the participants with their environment, it is easier to understand the participants' perspectives and intended meanings (Gray, 2013).

The epistemological conviction of this study is that of subjectivism. This implies that the grounds for rational belief are those of the community, in this case, the mathematics teachers in Thaba Nchu secondary schools. The researcher, together with the participants are interactively linked, leading to the data being created as the research study progresses (Gray, 2013). It is therefore a social form of subjectivism informing knowledge convictions. The study focused on the specific situations and environments of the participants, to understand their views and perspectives. The researcher's educational background and experiences shaped the interpretation of the participants' views and perspectives.

The population of this study

The population from which data could have been collected, consisted of all FET mathematics teachers in the wider Thaba Nchu district. The sample consisted of FET mathematics teachers from three schools in the mentioned district: School A (2 teachers), School B (2 teachers) and School C (1 teacher). The schools were chosen based on their overall functioning and performance: one school with above-average performance, one with average performance, and one school with below-average performance. For this study, the schools were named School A, School B and School C where the A, B and C were allocated to the schools according to the alphabetical order of their names, and not according to the school's performance. All three schools are from previously disadvantaged communities and are no-fee schools, which have to function on grants from the government. To protect the participants and keep high ethical standards, the participants were named A.1, A.2, B.1, B.2, and C.1. The letter A, B and C linked the participant to a school, and '1' and '2' indicated the number of participants from a school. The '1' and '2' were allocated to the participants according to the alphabetical order of their surnames. The chosen teachers felt confident enough to share their thoughts and feelings with the researcher and a relationship of trust and mutual respect was present. The sampling is therefore purposive (Plowright 2011).

Interviews as data collection method

Qualitative research aims to see the phenomenon under study through the eyes of the participant (Nieuwenhuis, 2007b). One of the methods commonly used in qualitative research is interviewing participants. Through the interviews, the researcher wanted to collect the teachers' opinions and suggestions, based on their perspectives of an ideal teaching environment, taking into consideration their circumstances, challenges and other factors influencing their teaching. It was also important to learn why they entertain their specific opinions about their teaching environment (Cai et al., 2009).

Discourse analysis focuses on making meaning of the written and spoken word to reveal why a situation is the way it is. To generate the discourse, semi-structured interviews were conducted with the chosen teachers from Schools A, B and C. In the semi-structured interviews, the teachers were asked to answer a pre-set list of questions informally. Using less structured interviews is an appropriate way of exploring interviewees' feelings (Plowright 2011:16). The questions were based on the categories of challenges identified through an in-depth literature review. The collected data were transcribed and translated as some of it were provided in a language other than English, namely Afrikaans. The information gathered from the interviews was used to get a deeper understanding of the teaching environments of mathematics teachers in Thaba Nchu secondary schools.

DATA PRESENTATION AND DISCUSSION

The interviews were transcribed and carefully coded, where after themes were identified. The themes were physical challenges, socio-economic challenges, teacher-faced challenges and management challenges. An analysis followed and are presented in the tables to follow. An interpretation accompanies each table.

The physical challenges identified by the five participants

Some of the challenges like resources and textbooks, were identified by more than one participant. A summary of the physical challenges identified by the participants is given in table 1.

TABLE 1: A summary of physical challenges identified by the five participants.

	A.1	A.2	B.1	B.2	C.1
Physical Challenges					
Resources	X	X	X	X	-
Textbooks	X	-	-	X	-
Class size	-	-	X	-	-
Technology usage in classroom and IBP	-	X	X	-	X
Physical state of classrooms	X	X	X	-	X
Water	-	-	X	-	-
Transport	-	-	X	-	X
Hostels	-	-	X	-	-
Sportsgrounds	-	-	-	X	X

From the information in the table, it is clear that physical aspects of teaching, in general, definitely create challenges for the teachers. These challenges influence the teaching and learning that take place in the classes.

A physical challenge like transport causes learners to miss classes, which results in them not gaining knowledge on a specific topic. Some learners do make an effort to get the work from either a class friend or the teacher, but not all of them care enough about their schoolwork to do this. When the learners write test, the marks are low and then they cannot understand why.

Another well-identified physical challenge is the physical state of classrooms. If the tables and chairs are broken or too few, learners need to share those available. Sometimes you will find three to four learners using one double table, or three learners sitting on two chairs. Such a situation does not contribute to successful teaching or learning. Learners are distracted easily, because they sit so close to each other. They can also talk while the teacher explains, causing the teacher to stop the lesson to get the learners focused again. If a teacher needs to do this a number of times during one lesson, he/she will lose time that will place a challenge on completing the term’s work on time for the test or exam.

Having a large number of learners in one class can create the same situation where learners need to share tables and chairs. Once learners must share furniture, maintaining discipline can become a problem. Teachers might need to address unwanted behaviour, like learners talking and not paying attention, more often. Valuable time is wasted every time a teacher needs to handle a disciplinary situation in class. Not only is the effectiveness of teaching negatively influenced, but learners are also deprived of their opportunities to learn. A large number of learners in the class brings a bigger workload for the teacher as well. The more learners a teacher has in a class; the bigger number of assessments must be marked by the teacher. To give individual attention to learners in a class with a large number of learners, tend sometimes to be impossible. Often the time available during a period to assist learners with individual problems is not enough. Teachers arrange extra classes after the end of the school day, to assist learners in this way.

Resources can be a physical challenge to teachers when resources like textbooks and photocopy machines are not available to address the need for it. If there are not enough textbooks in a class, learners need to share textbooks. In class, while the teacher is teaching, sharing of textbooks is possible. A problem arises when the teacher wants to give homework. If there is only one textbook between two or three learners, it happens that two of the learners do not do their homework with the excuse of not having a textbook. Teachers can solve this kind of problem by making copies of exercises and giving it to the learners without textbooks. If a photocopy machine is not available or not functioning or even if the school experience electricity problems, not having enough resources will stay a problem. Access to computers and the internet are valuable resources for teachers. Both can be used to prepare for lessons as well as to develop activities for classwork and set assessments.

Electronic devices, like laptops and data projectors, and other technological resources, like YouTube and mathematical software, can be worthwhile for teachers to use as part of their teaching strategy. Learners are fond of technology and their attention is easily grasped through the use of technology in a classroom. However, if the teacher must move from one class to another, it can be time-consuming and challenging to move the technological equipment with him.

Currently water, as a resource, is a sensitive topic in South Africa. For the last five years we are experiencing a shortage of water, due to droughts and low rainfall numbers. It is necessary that people adapt their ways of storing and using water. Not having running water in a household, is a challenge. The challenge is even bigger when a school does not have running water. Making a plan, to get a borehole shows that the school wants to eliminate water problems as far as possible.

Of the three schools I used in this study, only School A has a hostel. School C wants to build a hostel, but are waiting for approval from the necessary stakeholders. Having a hostel on the school grounds can address certain challenges like transport challenges. Learners who have to walk far to go to school or who must use public transport can apply to stay in the hostel. Learners staying in the hostel, will not be late for school in the morning. Not only will teachers be thankful for a lower late-coming number, but learners will also not miss school anymore. Attending school regularly, will assist learners in keeping up with the content explained in all of the classes and can even have a positive influence on results.

A part of a teacher's responsibility towards a learner, is the holistic development of the learner. Holistic development includes physical and social development and does not mean only cognitive development. One way in which physical development of a learner at a school can happen is by playing a sport like soccer or tennis. To be able to coach sport effectively, the school needs sport grounds and coaches. Sports grounds are not often seen at the Thaba Nchu schools. To let learners do some physical exercises, can also contribute to physical development. However, to let the learners do effective physical exercises, the school needs a space for this to happen.

To summarize the experiences of the teachers regarding physical challenges. Some physical challenges, like resources, was not experienced as a challenge by the participants, but rather as a positive aspect which needs to be nurtured. The other eight physical challenges identified by the participants, namely textbooks, class size, technology usage in classrooms, the physical state of the classrooms, water, transport, hostels and sports grounds, were experienced as challenges. These challenges need to be addressed in a way that it can be improved in the teaching environments of the participants as well as other teachers in similar schools or teaching environments.

The Socio-economic challenges identified by the five participants

All of the participants identified the support learners get from school as a positive aspect in their teaching environments. Learner attendance, parental involvement and parental support were the aspects that got responses from most of the participants. A summary of the identified socio-economic challenges is given in table 2.

TABLE 2: A summary of socio-economic challenges identified by the five participants.

	A.1	A.2	B.1	B.2	C.1
Socio-economic Challenges					
Learner support from school	X	X	X	X	X
Learner attendance	X	-	X	-	X
Parental involvement	X	X	-	X	X
Learners' attitude and action towards their parents	X	-	-	-	-
Parental support	X	X	X	-	-
Stationary	-	X	-	-	-
Poverty	-	-	X	-	X
Uniform	-	-	-	X	-
Discipline	-	-	-	X	-
Late-coming	-	-	-	X	-

'Learner support from school', 'stationary' and 'uniform' can all be directly linked to the 'poverty' aspect identified by the participants. All three the schools have a feeding scheme where learners are supplied with one meal every day. For the majority of the learners who only eat with the scheme, that meal is the only meal for the day. The poverty level of the communities surrounding the three schools, is high. Most of the learners come from low-income households where food is not always available. For this reason, all three the schools support the learners with food during the day. In extreme cases the school give the learners food to take home. Participant A.2 had many things to say about stationary being a socio-economic challenge and the influence, of learners not having stationary, on teaching and learning. If learners come from a low-income household where they do not have food to eat every day, chances are very small for those learners to have the necessary stationary. Learners adapted to this situation by sharing stationary, like pens and calculators, in class. Shared stationary leads to learners often asking for it, causing a disturbance in class for the teacher as well as for the other learners. Regular interruption of the teacher, while explaining, can cause the teaching and learning to be less effective. Learners, who do not have the needed stationary and who tries to keep up with the lesson, often have incomplete workbooks. Incomplete workbooks are, among other things, the result of corrections not done, because the learner do not have a pen or pencil to do corrections with. Another contribution to incomplete workbooks is when, graphs are not drawn, for example, because the learner do not get a chance to borrow a pencil to draw the graph. All the participants indicated that their learners have uniform to wear. However, they did not mention that the clothes are sometimes not the correct size, or that the clothes are not always in a good condition. Not having a warm enough uniform to wear during winter, together with a classroom with broken or no windows, mean that learners are not focused during lessons. It is difficult to concentrate and pay attention if one is cold and hungry. This situation leads to learners not understanding the lesson's content, not making notes in their workbooks and thus not getting good results in assessments.

Participant B.2 had a lot to share about late-coming being a socio-economic challenge. Late-coming and attendance of learners have similar consequences. Whether a learner come late or do not come to school at all for a day, they miss the content done in the classes. If they do not make an effort to get the work they missed, they have a gap in their knowledge. The schools do have plans in place to address late-coming problems, some of which very effective and some less effective. Effective plans lead to the decrease and even the elimination of late-coming, whereas less effective plans do not make a difference in the number of learners who come late for school.

Parental involvement and parental support work together. When parents or guardians support their children at home, by checking up on homework done or even assisting when the child has a problem with the content, chances are good that those parents will also attend parents' meetings and be involved at their children's school. A good relationship between teachers and the parents/guardians of a learner, means a faster solution to possible problems, due to an open communication channel. Learners, usually the ones who give discipline problems, tend to disrespect teachers and parents/guardians. Behaviour like disrespect is often a sign of learners having other underlying problems or challenges. With the necessary parental support, these problems can be spotted and addressed. Learners, no matter their age, do not always have the necessary skills to address emotional and development challenges. They need adults to assist and support them.

The participants' strong reaction to these challenges make me believe that they really feel the negative impact these challenges have on their teaching.

The teacher-faced challenges identified by the five participants

From all the categories discussed during the interviews, the teacher-faced challenges got the highest number or responses from the participants. A summary of the identified teacher-faced challenges is given in table 3.

TABLE 3: A summary of teacher-faced challenges identified by the five participants.

	A.1	A.2	B.1	B.2	C.1
Teacher-faced Challenges					
Teacher support	X	-	-	X	-
Attitudes of learners	X	-	-	X	-
Discipline	X	X	X	X	-
Crime/ violence	X	X	X	X	-
Motivation of learners	-	X	-	X	-
Teaching practice	-	X	-	-	-
Relationship with colleagues	-	X	-	-	-
Teacher class attendance	-	-	X	-	-
Teacher subject knowledge	-	-	X	-	-
Teacher qualifications	-	-	-	X	-
Support from DoE	-	-	-	X	-
Teacher attendance	-	-	-	X	-
Mathematics department	-	-	-	X	-
Class size	-	-	-	X	-
Language of teaching	-	-	-	X	X
Teachers' work ethics	-	-	-	-	X
Teachers' attitude	-	-	-	-	X

Many different aspects were identified as teacher-faced challenges, which gave the impression that this category is the biggest problem area for the participants.

The most problematic aspect seems to be the attitude of the learners. Both received responses were negative. The participants reacted strong to this aspect, indicating the seriousness of the influence of the negative, unconcerned attitude of learners on their teaching and what is happening in the classroom. Another rather alarming aspect is the low motivation level of learners. It is a challenging combination if learners have a negative attitude as well as low to no motivation, especially if the learners have the perception that mathematics is difficult and that they cannot do it.

Two other aspects mentioned by most of the participants, are discipline and crime/ violence. The discipline overall seems to be on a good level. It is only isolated cases or classes, sometimes even only a few learners, who needs to be disciplined. The rest of the learners seem to behave according to what is expected of them. Crime/ violence is not a problem in the schools, but definitely in the community. It did not feel as if it has an impact on any one of the participant's teaching.

From the responses it is clear that the participants are satisfied with the level of emotional and motivational support they get from their schools. The support is given by the principal as well as some of their colleagues.

The cooperation and relationships in the mathematics departments of the schools are overall positive and supportive. The participants felt a sense of belonging in the mathematics departments they form part of. The mathematics department needs to function as a team. Mathematics is a challenging subject to the majority of learners. It is therefore of utmost importance that the mathematics teachers plan together and find ways to assist and support each other.

Participant A.2 mentioned the importance of good preparation and planning of lessons. If teachers do not prepare properly, learners can lose confidence in the teacher. Especially if the teacher is disorganized and makes many mistakes during the lesson. Not being prepared, may have a negative impact on a teacher's confidence to teach the lesson as well.

A school which want to function optimally, needs to have unity among the staff members. A bad or strenuous relationship between staff members, can have a negative influence on a staff member's attitude and may even influence his/her teaching in a negative way.

Teachers must set an example to the learners by attending school every day and being in class, on time, every period. Teachers who are absent from school must be substituted by another teacher or a temporary teacher. Learners who sit without being constructively busy, tend to lose interest and start to behave in a way that can cause discipline problems. Teachers who take their time to get to class, not only create a foundation for learners to talk while they wait, but also loses valuable teaching time. With the pace against which the teachers must work to complete the term's content, teachers cannot afford to lose time.

Teacher subject knowledge and teacher qualifications often complements each other. Teachers with applicable qualifications to teach a subject, will have sufficient subject knowledge. Teachers should stay up to date with the changes in content, and if necessary improve their qualifications. Teachers can assist each other with the understanding of the content of a topic, if they are not totally comfortable with the topic.

The DoE is another source of support to teachers. Teachers and the DoE are role-players in the development of the learners. A relationship of trust and cooperation is needed between these two role-players.

The number of learners in a class can increase the number of assessments teachers must mark, as well as the maintaining of discipline in a class.

The language of teaching in South African schools is English. English is, in the majority of cases, the second or even third language of a learner. Not being the home language of a learner, can create difficulty in comprehension as well as ability to express his-/herself in English. Learners need to understand a question in order to answer the question. Teachers need to assist learner by teaching them only in English. This way the learners can learn the terminology of the subject as well as other language aspects.

To teach means you have an opportunity to make a difference in a child's life. What teachers tend to forget is that learners see teachers as their role-models and follow the example set by the teacher. If teachers have a negative attitude towards teaching or the subject, chance are good that this attitude with spill over to the learners. Teaching is a noble profession, where teachers do not only teach content, but also give care and life lessons which contribute to the holistic development of the learners.

The management challenges identified by the five participants

Management includes the principal, SGB, Senior Management Team (SMT) and Head of department (HOD). Aspects related to management by the SMT and communication between the SMT and the staff, are addressed in this section. A summary of the identified management challenges is given in table 4.

TABLE 4: A summary of management challenges identified by the five participants.

	A.1	**A.2**	**B.1**	**B.2**	**C.1**
Management Challenges					
Teacher support	X	X	-	X	-
Mathematics department	X	X	-	-	-
HOD	-	X	-	-	-
Communication	-	-	X	-	-
Principal	-	-	X	-	X
SGB	-	-	X	X	-
Extra classes	-	-	X	-	-
Management of resources	-	-	-	X	-
SMT	-	-	-	X	-

The general feeling towards support from the principal was very positive. From the responses it can be seen that management (principal, SMT, HOD and SGB) in the schools was supportive and structured in a way that the participants felt they have a back-up support system. Knowing there is a support system at their schools, influenced the participants' teaching environment in a positive way. However, participant B.1 had challenges with the principal and the communication between the principal and the rest of the SMT. Seeing that participant B.1 is a deputy-principal and thus part of the SMT, he experienced management from a different angle. The other participants are all post-level one teachers.

The management of the mathematics departments were perceived as positive by the participants. The HOD's of the respective schools' mathematics departments did not always feature as prominent in leading the department. However, the participants were not discouraged by a HOD not totally focused on mathematics. They focused on teamwork and good relationships.

The management of resources at the school was only mentioned by one participant. For resources to be available without difficulty, good management of these resources has to be in place.

Participant B.1 mentioned extra classes as negative aspect of management. He felt that if the teachers are in their classes, as they are expected to be, extra classes would be unnecessary. It is important that the management of the school paid attention to aspects of teaching like teacher attendance in class. If teachers were in every class according to their time table, challenges like not having enough time to complete the term's subject content and discipline problems, would not often be obstacles. Extra classes could then be arranged to assist learners with individual content problems and not to complete the terms' subject content.

CONCLUSION

The interviews with the participants did not only give an idea of how they experience teaching in their immediate environment, but also gave a look into their hearts. They have the learners' best interest at heart and they have a passion for the subject, the learners and the profession. Sometimes they experience challenges that cause obstacles in their way, but they are still happy and want to make a change in learners' lives.

In this study data was collected from doing interviews with the participants. The data was analysed, interpreted and discussed resulting in a way that the researcher can have a deeper understanding of how the participants' experience their teaching environments in Thaba Nchu schools. Thus answering the research question "How do practicing mathematics teachers in Thaba Nchu schools experience their teaching environments?".

The individual participants identified different aspects in each group as challenges and as positive aspects. The unique experiences of the participants are valuable information which forms part of a bigger research study done to answer further research questions. These research questions and the research conducted to answer them, will be addressed in future publications.

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STUDENTS' CONCEPTUAL UNDERSTANDING OF CUBIC FUNCTIONS IN DIFFERENTIAL CALCULUS

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The exploration of properties of functions and representation of mathematical situations is usually facilitated through the knowledge of coordinate geometry. This paper reports on a study conducted with forty-two grade 12 students in the Mt Ayliff district of the Eastern Cape Province in South Africa on their conceptual understanding of cubic functions in differential calculus. The study was underpinned by the variation theory that requires students to discern the critical features of an object in order to understand conceptually. This was a case study design from an interpretive paradigm perspective in which qualitative methods were used to interpret students' responses in a questionnaire administered on cubic functions. An instructional design that followed the initial genetic decomposition of cubic function representations guided preliminary two weeks one hour lessons conducted with the students. Questionnaires with questions on the understanding of cubic functions were administered to the students to respond to the written tasks. Semi-structured interviews were then conducted with a purposefully selected sample of three students on the basis of their responses in the questionnaire. Results indicated that the multiple procedural steps required for the representation of cubic functions call for students to have a fusion understanding of the concept.

Keywords: conceptual understanding, cubic functions, differential calculus

INTRODUCTION

In the physical world, functions usually represent relationships between physical variables. For example, the level of hunger may be a function of the signal of the time of the day as indicated by internal biological rhythms. In mathematics functions are defined in terms of symbols sometimes indicating a one-to-one correspondence to physical variables (Albus, 1981), or a directional relationship between cause and effect that is sometimes expressed as a graph. Students in lower grades are usually introduced to functions through plotting of points represented by coordinates extracted from tables although the accuracy of continuous functions depends on some discreet points chosen in the table. In the senior phase, students are expected to represent linear functions algebraically and graphically. Later, in the Further Education Training (FET) phase, students draw linear functions using other methods like the location of the x and y-intercepts or the gradient calculations. In addition, they are then introduced to quadratic functions. Finally, multidimensional functions such as cubic functions that cannot be drawn by table coordinate look-ups are interrogated. A cubic function is any function of a polynomial whose greatest exponent is of the third power of x. For a student to be able to solve cubic functions, he/she needs to master specific mathematical skills including the knowledge and applications of the factor theorem, algebraic manipulations of algebraic functions together with techniques using derivatives in differential calculus.

Hourigan, Leavy, & McMahon (2012) suggest that the construction of the knowledge of cubic graphs has its roots in the understanding of functions, a topic that provides opportunities for students to develop algebraic thinking skills and to generalize findings on calculations. In South Africa, cubic functions are only taught at grade 12 level. Most mathematics teachers at that level dwell on merely assisting their students to obtain the subject pass mark, than steering fundamental ideas and meanings for conceptual understanding. Thompson (2008) further distinguishes an idea as something that has context and meaning, is generalizable, and has the potential of being foundational for other ideas and ways of thinking. The author defines meaning as that which comes to mind to make a word, phrase, observation, sensible and comprehensible to the student. In this paper, I explore the conceptual understanding of cubic functions in differential calculus through the variation theory to access whether they displayed an in depth meaning into the understanding of those functions. The considerable evidence within South African schools, pointing to the challenges faced by students' representations of cubic functions is not repeated in this article. Such evidence is available in abundance (Brijlall & Ndlovu (2013), Postelnicu, (2011), & Maharaj (2010)). What is lacking in literature is the exploration of the conceptual understanding of cubic functions in differential calculus by grade 12 mathematics students. I argue in this paper that the learning of cubic functions with meaning occurs when the students can fuse several aspects of the concept with simultaneous discernment of all of the critical features together with the relationships and apply that knowledge to solve new connected problems. This paper responds to the question: 'How do students construct mathematical knowledge when learning cubic functions in differential calculus?' In the first part of this article, I venture into the historical origins of cubic functions and their applications in mathematics. Next I provide the genetic decomposition necessary for understanding cubic functions. And finally I conclude by presenting suggested compositions of the critical features of cubic functions.

LITERATURE REVIEW

Leong (2013) notes that students' difficulty in understanding the concept of function stems from its dual nature. A function can be either be understood structurally, as an object and operationally, or as a process. According to Brinkmann, Goedgebeur, & Van Cleemput (2013) a cubic function can be represented as a simple graph in which each vertex is adjacent to three other vertices. This would denote a structural representation. However, cubic graphs have long received much attention from both mathematicians and chemists. Greenlaw and Petreschi (1995) assert that cubic graphs history connects to three classical topics in graph theory, (i) the maximum matching problem, (ii) the four color problem and (iii) the theory of planar graphs. Cubic functions further play a role in testing matching algorithms, nonetheless grade 12 students are only introduced to mechanisms of drawing and interpretation of those functions. Hence the aforementioned concepts that constitute the historical values of cubic graphs will not be elaborated on in this article.

Yamada (2000) conjects that the understanding of functions does not appear to be easy, because of the diversity of representations associated with the concept. In addition, the difficulties presented in the processes of articulating the appropriate systems of representation involved in problem solving are also challenging for students. According to Evangelidou, Spyrou, Elia, & Gagatsis (2004) a function is usually connected with a type of diagram, either a Cartesian graph or a mapping diagram. Students are introduced to the plotting of graphs on the Cartesian plane, representations and descriptions of situations in algebraic language, formulae, expressions, equations and graphs as early as grade 8 in the senior phase. Yet, it has been observed that they join the Further Education and Training band with inadequate understanding of how to distinguish the critical features of each of the functions. In particular, students fail to understand the processes involved and the difference between linear, parabolic, hyperbolic and cubic functions.

For students to be able to represent cubic graphs they need to be able to construct mental representations of : (i) finding of intercepts, (ii) use differentiation to find stationery points and substitution thereof, (iii) either of both local minima or maxima, (iv) points of inflexion if applicable, (v) drawing the shape of the graph and (vi) variations on increase and decrease of the cubic graph. Dubinsky (1991) refers to the mental constructs that a student must make in order to understand a concept as the genetic decomposition. The genetic decomposition is a set of mental constructs which might describe how the concept can develop in the mind of an individual. The afore-mentioned genetic decomposition for cubic functions guided the instructional design and activities chosen to equip the students with the knowledge of cubic functions prior to the administration of questionnaires. In this article I claim that a student has to discern all the critical features, separate them and fuse them to apply to the constructs of understanding the cubic functions. The essence of inadequate preparation of students in previous grades seems to affect their understanding of mental constructs that serve as basis to connect to higher critical knowledge of functions.

THEORETICAL FRAMEWORK

Magill (1989) asserts that students' performance is an observable behavior when learning has taken place as an internal phenomenon. I argue in this paper that conceptual understanding is therefore an internal construction of a meaning attached to a concept when learning has been inferred from an observable behavior. The depth in conceptual understanding of cubic functions is determined by how students connect their understanding of factorization in relation to finding and applying their knowledge of derivatives. On the contrary understanding a concept in this article is explained through the variation theory.

Leung (2012) describes variation in terms of what changes, what stays constant and how students discern underlying rules in the process of learning a concept. The object of learning in this article relates to the students' understanding of the representation of cubic functions. This article strives to isolate the critical features that must be discerned in order to constitute the meaning aimed for in learning cubic functions. Marton (2009) proposed four kinds of awareness brought about by different patterns of variation. Hence the researchers define the following variation concepts based and described on those given by (Leung, 2012), Mohlolo (2013):

Contrast: awareness that pre-supposes that to know what something is, one has to know what it is not, that is, to discern or learn whether something satisfies a certain condition or not (Leung, 2012). **Separation**: focus on certain features that are critical and draw students' attention selectively to the critical aspects of the object of learning. Hence, according to Leung (2012), **Generalization** is a verification which separates the specific or particular from the general to what is applicable to all, and **Fusion** takes place when the student integrates critical features and focus on different aspects of an object of learning at the same time.

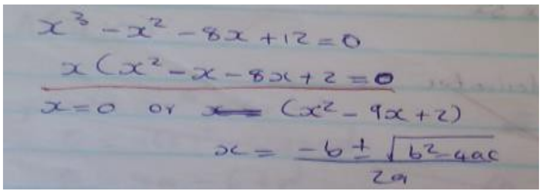
In this article, I argue that although the students join grade 12 underprepared to understand relevant mathematics concepts, the variation theory guided by instructional design based on the proposed initial genetic decomposition, can assist the students to understand cubic functions. This theory was chosen as it emphasizes variation as a necessary condition for learners to be able to discern new aspects of an object of learning, which in this case is cubic functions. By using variation and invariance within and between examples, students engage with mathematical structure of the cubic function.

METHODOLOGY

The study adopted the interpretive research paradigm. Students were initially exposed to instructional design using activities that intended to equip them with the conceptual understanding of cubic functions as prescribed by the genetic decomposition of the concept. Although a questionnaire was administered to the students as a class activity, I was not interested in whether the responses were right or wrong. Of importance was how the responses revealed conceptual understanding and the interpretation of cubic functions. A multiple case study design was therefore followed to reach to the depth of students' interpretations of the presented tasks. Contrary to Johnson & Christensen (2014) emphasis that questionnaires are used to obtain information about the thoughts, feelings, attitudes, beliefs, values, perceptions, personality, and behavioral intentions of research participants, the intention of the questionnaire was to establish how students interpreted and responded to the tasks. Forty-two grade 12 students participated in this qualitative study. A purposive sample of three students (S12, S22, and S41) were selected on the bases of their responses and interviewed individually for approximately thirty minutes. The idea was to work through the logic behind what the students could be able to do and write after they had been engaged with cubic functions through the theory of variation. Critically, the responses were examined for students' ability to label the intercepts, points of inflection, the minima and maxima and the shape of the drawing produced for the cubic functions. In this verification and conjecture-making activity, students had to describe the general steps they followed in order to draw the cubic functions. In addition to the tasks, semi-structured interviews were conducted to substantiate how students constructed meaning for understanding the cubic functions. These interviews were more valuable than the written assessment instruments because one student could have displayed correct written work in the transcript while the explanations revealed little understanding and vice versa. A full range of understanding was obtained by selecting students who gave correct, partially correct and incorrect responses on the questionnaires.

DATA REPRESENTATION, ANALYSIS, FINDINGS AND DISCUSSIONS

After the administration of the questionnaires with tasks to the forty-two students, they were coded as responses, S1 for the first student up to S42 for the last one. Some patterns identified in the responses were used to classify the responses into three categories. Category 1-3 was made of the responses that displayed full understanding, partial understanding and no understanding respectively. Lastly the responses were broken down into segments of knowledge revealed in the students' explanations during the semi-structured interviews. I therefore present the response given to the first question followed by the explanations given by each of the participants during the semi- structured interviews. Question 1 required the students to factorize the equation fully. S12's response is displayed in Extract 1.



Extract 1: S12's response

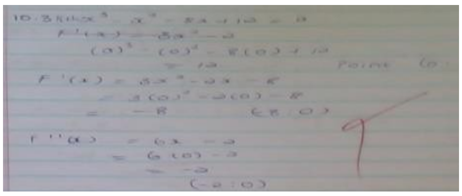
During interviews, the student was requested to explain how she found the *x*- intercepts? S12: *By letting y or f(x) to be equal to Zero, and solving for x*

Interviewer: 'How do we solve for *x* then?'

S12*: 'By either taking, eh a common factor or factorizing*.'

Interviewer: ‘Let’s look on your answer here (pointing on the answer script of the student), the first step and the second step you took x as a common factor. When should we take a common factor? Or what prompted you to take x as a common factor here?’
S12: ‘*when there is, mmm, x appearing more than once on the equation*. ‘
Interviewer: ‘What is a common factor then?
S12: ‘*A number that can multiply two or more numbers separated by a plus or minus sign*. ‘
S12: ‘*Hmm, because of 12 has no x. So, it does not have a common factor’*.
Interviewer: ‘Why didn’t you use that method in your answer? Can we use the quadratic formula to solve this? Why? What make this different from quadratic equations?’
S12: ‘*I, I thought it needs a common factor, but, ehh, now I see that 12 has no x, and now I see that we cannot use the quadratic formula. It is a cubic equation with x to the power 3. Quadratic equation has x square on it as the higher exponents, let me now go and try solving it.*

In this excerpt, it becomes evident that the contrast as posited by Leung (2012) has helped the student in differentiating when to take out a common factor by contrasting an equation that has a multiple of *x* throughout the equation and that does not have a multiple of *x* throughout the equation. Initially, it appears that the student could not differentiate between (i) cubic and quadratic equations, (ii) factorization strategies, and or (iii) simplification of algebraic expressions. The theoretical underpinnings of this study appear to have at least assisted the student to contrast and to differentiate between the different types of algebraic equations together with how to solve them. The researcher varied the equations systematically to draw the attention of the student to the separation of the critical features of both cubic and quadratic. The student was also led to contrast and separate factorization techniques that were suitable for the different equations. This enabled the student to make the link, to differentiate and to solve for *x* in the cubic equation, a requirement for sketching a cubic graph. The second question required the students to find the stationery points for the function. Extract 2 represents S22’s response to the question.



Extract 2: S22’s response

During semi-structured interviews, the student was asked to describe the steps he followed to establish the stationery points.

S22: '*I thought that I must get the coordinates of the turning points of the graph. Firstly, ehh I think I need to differentiate the equation. And, mmh….. substitute by zero.'*

Interviewer: 'why do you substitute by zero?'

S22: '*To get y- value, Mevrou'*.

Interviewer: 'Which y-value on the graph? '

S22: '*The turning point y-value'*

Interviewer: 'Do you have the x – values for the turning points?'

S22: *hmmm, no, eh…..*

Interviewer: 'how do we get the x-values for stationery point?'

S22: '*we find differentiate first'*.

Interviewer: 'then?'

What?

S22 (Silent)

Interviewer: 'How do you solve for x in a quadratic equation?'

S22: '*By either factorizing or taking out a common factor.'*

Interviewer: ' when do we get the root of an equation?'

For example given the following:

$f(x) = x^2 - 3x + 2$, what must be done to get the roots of the equation ?

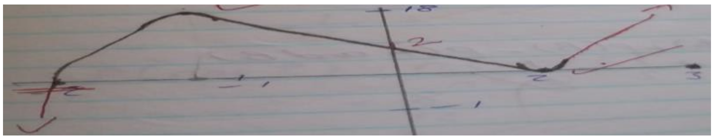
S22: '*Mevrou, what are roots now? We want to find the values of x neh? I think ehh, f(x) should be equal to 0'*

Interviewer: 'what about solving for x during stationery points?'

S22: '*oh, we should we I'll finish it up!* search *for stationery points also, now I understand. Ok leave it sir now*

Though the student managed to differentiate during the writing process, further steps indicate that he was unable to find the stationery points or unable to know what should follow after differentiating. Language also seemed to the barrier for this student since he could not associate the roots of an equation and values of the function. This was also evident in the case of stationery points.

Also, it was revealed that though the student was able to differentiate, he was unable to factorize the differentiated sum. Very often the student rushes towards using the quadratic formula in order to find the factors, especially with equations that have a coefficient of 2. This deprives them with an opportunity to apply critical thinking skills to find the correct factors through trial and error. The third question required the students to find the points of inflection and determine the local minima or maxima of the function representation together with whether the shape of the graph will be concave up or concave down. Extract 3 represents the response given by S36 in this question.



Extract 3: S36's work

Although the student correctly represented the graph, during interviews he was unable to justify why the shape of the graph was concave. He also could not explain or predict the number of stationery points for this graph. Procedurally he followed all the steps correctly and hence was able to represent the function graphically and could differentiate between a local minimum and maximum. However, the discernment of the critical features of the graph were not clear and hence the students demonstrated separate chunks of the drawing process but could not fuse them. I argue that given another cubic function, the student might not be able to sense if incorrect factors that may lead to an incorrect graph representation emerges.

CONCLUSION

Indeed, for the conceptual understanding of cubic functions, it is essential students integrate the critical features of cubic functions, while fusing by connecting variation experiences gained in previous algebraic manipulations. Those include identification and meaning of the (i) x -intercept, (ii) y-intercepts, (iii) stationery points, (iv) shape of the graph (v) factorization of the given expression, (vi) differentiation the function and (viii) fusion of all those to represent the function graphically. Substitution of the value of x into the original equation to get the value of $f(x)$ is a necessary skill. The findings in this study indicate that it becomes important to afford the students an opportunity to contrast what cubic functions are and what they are not. Most importantly, the students to separate the critical features of solution of cubic functions and strengthen their factorization skills with regards to quadratic and cubic functions. It is important to note that besides synchronic simultaneity when students focus on different aspects of an object of learning at the same time, diachronic simultaneity also plays a critical role in fusion by connecting variation experiences gained in previous and present interactions (Leung, 2012). Multiple procedural steps required for the representation of cubic functions call for students to have a fusion understanding of the concept. What is and what is not made possible to learn, seen from a variation theory perspective, is often analyzed from classroom data in which the interaction between the teacher and the students in regard to the cubic functions in focus.

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A STORY OF ENCOUNTERS OF LEARNERS WITH ASPECTS OF INTEGER ARITHMETIC EMBEDDED IN DEEPENING MATHEMATICAL THINKING TASKS

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I report on the implementation of deepening mathematical thinking tasks dealing with integers in two grade 8 classes. The tasks are part of a proposed approach to deal with the forget question. Educational story telling is the paradigm underpinning the article. The stories are the reports of the implementation, with me as the 'teacher', given to members of a development and research group concerned with the enhancement of achievement in school mathematics. No conclusions are drawn and recommendations or prescriptions, other than for the project group, are not given. As is the case with most stories, it is left to the reader to construct their own interpretations as they feel fit.

INTRODUCTION

Reading the *Overstory* (Powers, 2018) I came across an incident where there is a discussion about convincing through 'research' outcomes. The novel is, in part, about activists and their actions against deforestation. All the activists had some experiences with the value of trees in their lives. One of them was an actuarial science student who dropped out of her studies. Her partner was an artist. During one of the actions the two camp out in a tree to prevent it from being felled. A doctoral student in psychology researching the "personality profiles of environmental activists" (p. 319) joined them in the tree with the purpose of interviewing them. During their interactions one of the activists asked the doctoral student: "How do we convince people that we're right?" The student responded, "The best arguments in the world won't change a person's mind. The only thing that can do that is a good story" (p. 336). This utterance prompted me to report 'research' in the paradigm of educational story telling. I delve into the value of stories, the use of stories in mathematics teaching, stories about the pursuit to solve an unsolved mathematical problem and the quest for stories as a report format for 'research' endeavours which put a high priority on the developmental aspects of research and development. The next section provides the background and context which is followed by a description of the approach followed. The penultimate section presents the stories of the implementation of the deepening mathematical thinking tasks. Closing remarks are given in the final section.

STORIES AND STORIES IN EDUCATION

Brown (2017) relates the value of stories as inspiration for writing ideas emanating from her research. Referring to J. K. Rowling of Harry Potter fame, Brown (2017, p. 4, italics in original) asserts "I imagine her (Rowling) telling me: *New worlds are important, but you can't just describe them. Give us the stories that make up that* *universe. No matter how wild and weird the new world might be..."* The message coming to the fore, to me, from Brown's assertion is that stories about particular universes must be told and in development and research reporting one should not steer away from doing it through stories.

There are many uses of stories in mathematics teaching. Noddings (1997) demarcates five different categories of stories that can be used in mathematics teaching. These are: historical and biographical stories; novels in which mathematics feature prominently; personal stories, humorous stories and expositions about the psychology of learning. She warns against the use of these stories as merely time fillers and opines, "often stories turn out to be more effective than arguments and explanations" (Noddings, 1997, p. 1). As an example of a novel in which mathematics features prominently the one by Doxiadis (2000), dealing with search for a solution of Goldbach's conjecture forms the background, is of the latest ones I have read. In the acclaims section of the book the late Sir Michael Atiyah, a field medalist, has this to say about the book "It is brilliantly written – a mathematical detective story of great charm…" (Doxiadis, 2000, no page number). A review of the book in a publication of the American Mathematical Society states "one of the strengths of [the book] is that [it is contend] to stay on the level of good storytelling rather than straining for something more highfalutin." (Jackson, 2000, p. 1274).

Having given a brief rendition of the value of stories above, it is no wonder that Denny (1978) placed story telling alongside the qualitative research methods of ethnology, ethnography and case study. According to him "a story documents a given milieu in an attempt to communicate the general spirit of things. The story need not test theory; need not be complete; and it need not be robust in either time or depth…The story is the first cut at understanding enough to see if a case study is worth doing." (Denny, 1978, p. 2). He further expands on fieldwork, data gathering and the reporting of the 'research' from a story telling perspective. An example of Denny's application of storytelling is his evaluation report on science (including mathematics) teaching and learning in some United States of America's public schools (Denny, 1977).

Observable from Denny's work is that the reporting of the 'research' in a journalistic manner is imperative in educational story telling. This reporting is at the heart of developmental research to impact the development of teaching and stated by Freudenthal (1991, p. 161) as

> Experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience.

The issue being 'journalistically reported on here is "what do learners produce when they encounter deepening mathematical thinking tasks with me as the 'teacher' dealing with the task and their teacher afterwards, during the same period, continuing with normal teaching?"

BACKGROUND AND CONTEXT

Denny (1977) suggests that for educational story telling as much as possible background and context of the site of the story need to be provided. Ostensibly such information contributes to Freudenthal's (1991) becoming part of "their own experience".

The learners involved were from two different grade 8 classes of the same school. The school is in one of the fruit-growing districts in the Western Cape about 175 kilometres from Cape Town. It is an all-grade school—grade R to grade 12. The school is a quintile 2, non-fee paying one and part of the school-feeding scheme of the state. Observable is that the school grounds are reasonably clean. The school prides herself in the above 90% National Senior Certificate (NSC) pass rate of the school. This is deemed comparable with pass rates of schools serving learners from high socio-economic status backgrounds in the district. The 2018 performance in the NSC examination for selected subjects published by the Department of Basic Education (2019) is given in Table 1.

TABLE 1: Pass rates in selected subjects

(Number of learners passing subject at 30%)/Total Wrote							
Business Studies	English First Additional Language	Geography	History	Life Sciences	Mathematical Literacy	Mathematics	Physical Science
35/97	101/101	43/68	36/51	11/16	42/93	5/11	6/7

The school is participating in a Continuing Professional Development (CPD) Project for mathematics teachers. The ultimate aim of the project is to increase the number of learners offering Mathematics as an examination subject in the NSC examination and to enhance the quality of the passes. The data of the school's performance in the NSC Mathematics and Mathematical Literacy examinations since the start of the school's participation in the project are given in Table 2. At a global school level, the Table 2 also indicates that from 2016 the number of learners writing the NSC examination increased from 61 to 104 given that all learners have to offer either Mathematics or Mathematical Literacy for this examination.

TABLE 2: Performance in the NSC Mathematics (MTH) and Mathematical Literacy (ML) examinations

Year	2016		2017		2018	
	MTH	ML	MTH	ML	MTH	ML
Wrote	6	55	7	52	11	93
Number passed at 30%	6	45	6	40	5	42
Percentage Pass at 30%	100	81,8	85,7	76,9%	45,5	45,2
Average % Examination Score	36	33	38,4	38,2	37,8	30,6

Although the overall aim of the CPD project is anchored around enhancing achievement in mathematics in NSC mathematics examination. It works, where possible, with a critical mass of members of a school's mathematics department to establish a particular pedagogical culture for teaching of mathematics across the project schools. This culture is called productive practice. Productive practice has links with Freudenthal's (1991, p. 114) idea of "training integrated with insightful learning" but includes development of procedural fluency driven by two cognitive levels, knowledge and routine procedures. Productive practice focuses on work that has already been taught. One of the purposes of productive practice is to address the forget question. The forget question is the phenomenon that immediately after a construct, such as a procedure to add integers, has been taught, learners can deal with exercises and tasks related to the construct. After some time they experience difficulties with the exercises and tasks related to the construct. As articulated by a teacher in Denny's (1977, p. 1-37) study "[they] know one thing perfect in math one day and the next it is gone ...I mean gone."

Two kinds of tasks characterizes productive practice. The one is spiral revision focusing on consolidation of mathematical procedures and concepts that are practiced in a distributed manner (see for example, May (2014) and Okitowamba, Julie & Mbekwa (2018)). The second one is termed Deepening Mathematical Thinking (DMT) which is the fostering learners' engagement with mathematics through exposure to teaching and learning environments and challenging mathematical activities that develop:

1. Deeper understanding of mathematical concepts and procedures.
2. A disposition of productive struggle with mathematics.
3. The ability to explain and justify solutions.
4. Reflection of their ways of working.

A major difference between other approaches that aim to develop the mathematical fluencies and processes referred to above is that there is no insistence that teachers drastically change their developed ways of teaching. This is based on the project's acceptance that every teacher has developed, through experience, some personal mathematical pedagogical philosophy to realize the goals she or he is striving for with teaching. A second reason for following a route of minimal insistence that teachers should change their personal ways of teaching is that suggestions for teaching must be deemed to be ecologically relevant. Ecological relevance refers to teachers deeming the implementation of ideas offered during workshops and institutes as doable within the functioning milieu of their schools and classrooms with their varying demands. This is no different from the expectations of medical practitioners who attend continuing professional development courses and "…tend towards… that which is meaningful to their life situation with immediacy of application" (Gibbs, Brigden & Hellenberg, 2005, p. 5). A final reason for minimal insistence and suggestions for teachers implementing major redirections in their practice is linked to the idea that consistently incorporating small steps in practices have demonstrated to lead to improved outcomes. Wiliam (2016, p. 174) relates how the golfer, Tiger Woods, improved his swing by consistently making small changes to it and recommends that

> …teachers have to make small, incremental, evolutionary changes to their practice. Teachers have to change their practice while keeping all other routines functioning normally. For this reason, we recommend that teachers only change only a small number—ideally one or two, and certainly no more than three—aspects of their practice at any one time.

A primary reason for this recommendation is that teachers, and for that matter anyone involved in teaching in any area have developed repertoires of pedagogical practices. These practices have become habituated. In general changing habits is difficult and hard. It is no different for redirecting one's pedagogical practices. Therefore as a "small step" teachers are merely encouraged to incorporate tasks underpinned by productive practice in their normal teaching. Figure 1 presents the guiding suggestions for incorporating 10—12 minute productive practice task in a normal lesson.

EXAMPLE of INCORPORATING PRODUCTIVE PRACTICE in a NORMAL LESSON

The topic being taught is "Algebraic expressions". Completed topics are: Real numbers (rational and irrational numbers), surds, multiplication of a binomial by a trinomial and the factorization of the difference of two squares. The section being taught is the factorization of trinomials. In the first lesson learners were given the first 4 problems from the exercises section of the textbook and another 4 problems for homework. The next lesson period can then proceed as follows:

$\pm$10 minutes	Productive practice set on previously taught topic, say set 1 on real numbers
$\pm$10 minutes	Checking and marking of homework
Rest of period	Limited number of problems from the exercises section in the textbook on the factorization of trinomials.

Figure 1: Suggestions for incorporating productive practice in normal teaching

APPROACH

Nearly every week in the book review section of the *Sunday Time's Lifestyle Magazine* there is an interview with an author (storyteller). One of the questions they are invariably asked is about the research they conducted for their stories. Dependent on the genre—historical, crime, science fiction, romance, travelogue, etc.—of the book, the authors would reveal the particular ways of gaining necessary ideas for their stories. Interestingly, they seldom speak about research methods. They would, say, describe sites they have visited and resources they have consulted to collect necessary information for their stories. The approach used in this article is somewhat similar. Thus I am heeding Denny's advice about "Looking *at* something rather than looking *for* something…" (Denny, 1978, p. 13). In normal educational research parlance, I was a participant observer. My "observation" notes were photographs of learners' work and whiteboard writings, my post-lesson scribbled notes and audio-recordings to primarily capture my utterings and those of one or two groups of learners. An example of scribbled notes is given in Figure 2. I do not find myself as disciplined as Denny (1977, p. 2) who in "In the course of five weeks…filled twenty spiral hip-pocket notebooks", when he was conducting an evaluation study.

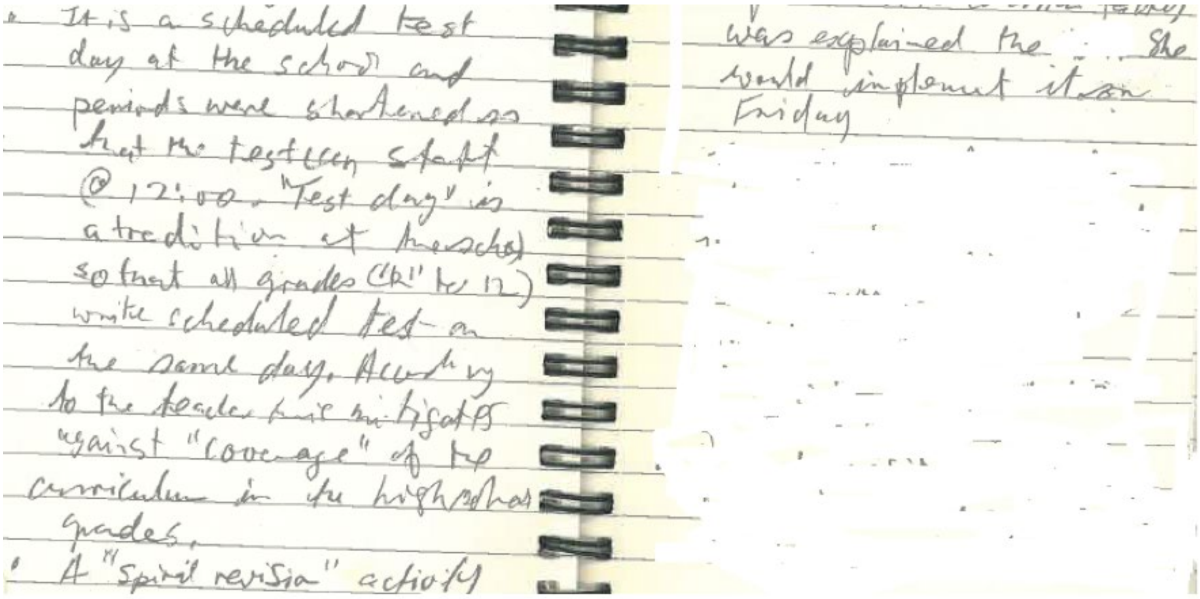


Figure 2: Example of scribbled notes

I carry two digital recorders with me and they are normally switched on when I start an activity and switched off at the end of the activity. These sources were used to construct the story that is normally in the form of a report distributed to the other members of the CPD development and group.

THE "STORY" REPORTS

The first report is about learners' encounter with a task involving addition integers and the description of an observed pattern from the obtained responses. I had two activities from the project's battery of activities. One I used and the other was given to the teacher to use during a later lesson. Forty copies of each activity were made. This was based on that the class would be about 40 learners. The activity sheet (see Figure 3 below) was handed out to the learners and I gave a brief explanation of what must be done. I observed that about two learners responded to 1.1 but others displayed uncertainty as to what they must do.

Set 1

1. Indicate the answers below on the given open numberline.

0

1.1) 2+3=A

1.2) 2+2=B

1.3) 2+1= C

1.4) 2+0=D

1.5) 2+(-1)= E

1.6) 2+(-2)=F

1.7) 2+(-3)=G

1.8) 2+(-4)=H

What is the pattern that you see above? Write it down in your own words.

Figure 3: Grade 8 “Spiral Revision” Activity on Integers

I intervened and dealt with 1.1 and 1.2, using the whiteboard to illustrate to the class what they should do. During this exposition learners were called to the front and place the answers on the number line as shown in Figure 4.

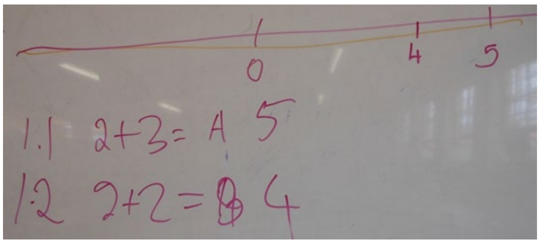
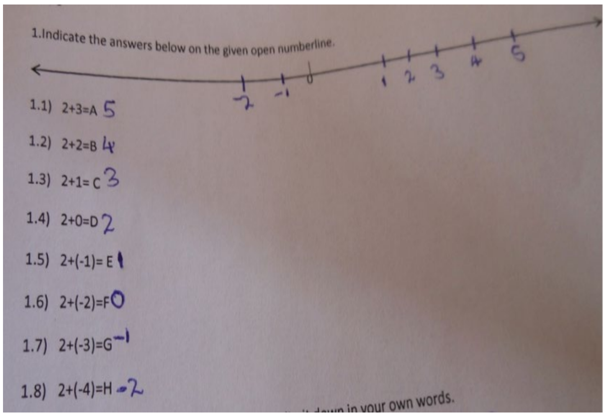


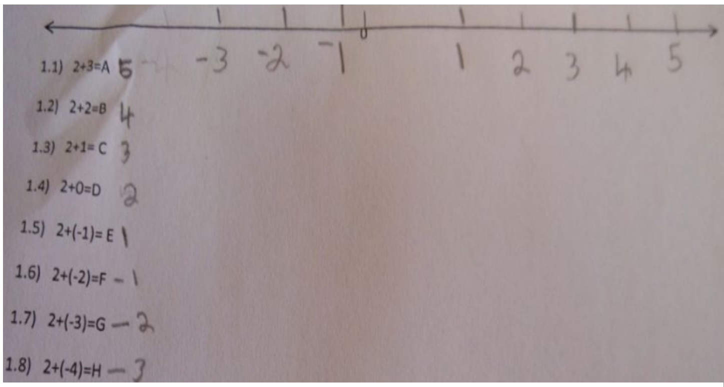
Figure 4: Whiteboard representation produced by facilitator and learners

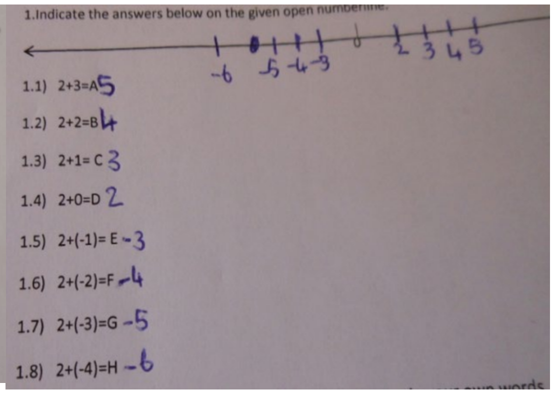
Learners also wanted to know whether they had to write their names on the activity sheets (My mental note: *Is this indicative that learners viewed the activity as a “test” that will be marked?*) I responded that it was not necessary and that they can paste it in their exercise books. At the start learners wanted to transfer the problems to their exercise books (My mental note: *Is this indicative of an encultured ‘habit’ of first writing down the problems or work on the chalk-/white board in exercise books?)*

Learners proceeded with the rest of the activity. Randomly selected shots representative of learners’ responses to the “calculational” part of the task are given in Figures 5(a) to 5(c).



5(a)





5(b) **5(c)**

Figure 5(a) – 5(c): Learner's response to "calculational" part of the task

The activity also required learners to identify a possible pattern linked to their responses. Not all learners completed this part of the activity in the allotted time. An example of the most typical response to the "description" part of the activity is given in Figure 6.

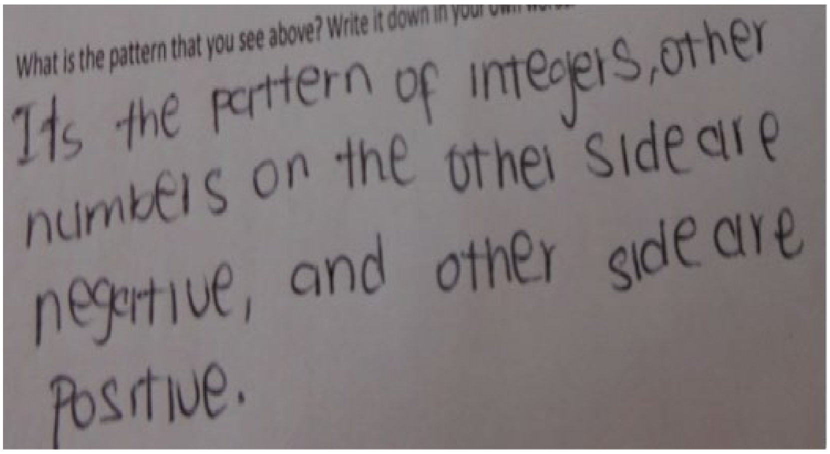


Figure 6: Most typical response to the "description" part of task

The "productive practice" activity was concluded by me telling the learners to, at a convenient time, compare their responses to check what possible correct answers are.

The activity lasted about 10 minutes. However, the total time since the time-tabled scheduled commencement of the period (after the break) was about 20 minutes—time was taken up by learners settling down, arriving late in class, introduction of a stranger (me) and other matters teachers normally settle before actual teaching starts.

At the end of the activity, the teacher continued with his planned classroom work. The learners had to complete the compulsory investigation that learners had to do as part of their term assessment pieces.

On personally reflecting on the implementation episode I generated the following ideas:

1. The existence of an ill-formed calculational rule regarding negative integers—"work with the numbers without the signs, perform an operation (addition in this case) and attached the sign of the number with the greatest absolute value as in "2 + (-4) = - 6" in Figure 4(c) above. This is also linked to the unobtrusive switch from the unary meaning of the minus sign (-) to the binary meaning. A question that arose from this observation: Can an activity or representation be designed in order for learners to distinguish between these two meanings?
2. Unintended consequences emerge with pattern-like activities. In 4(b) above it appears that the pattern of decreasing integers is identified. When 2+ (-2) is reached it assumed that the number following 1 in 2 + (-1) must be -1 and this reasoning is followed through to obtain the responses -2 and -3. What activities can be designed to address this issue?
3. Although the learners could articulate a pattern as shown in Figure 6, the description is at a visual surface level more linked to the number line. Learners tend to ignore that the first addend, 2, was kept constant. What kinds of activities are needed to develop learners' competencies to focus on more salient aspects of a pattern? How are their competencies to write up observations enhanced?

I implemented the second activity after two weeks. As mentioned above, during the first visit I did one activity and requested the teacher to implement a second activity at a suitable time. When I enquired about the implementation, the response was it could not be implemented due to the term tests that were written. The teacher was again requested to implement it at a convenient time and should consider reporting the experiences with it at an upcoming institute. There were 27 grade 8 learners present. Four grade 7 learners joined the class. This is a practice at the school to deal with absent teachers—learners from the classes of the absentee teacher are divided into groups who are assigned to different teachers for supervision.

I attempted to be more disciplined to monitor the actual time by switching the digital recorders on when I started the activity.

The activity was an ASN one given Figure 7.

Mathematical Statements	Always True (A)	Sometimes True (S)	Never True (N)
When a number is subtracted from 5 the answer is less than 5			

Figure 7: Task given to learners

The learners were given opportunity to read the activity. After they had perused it, the following transaction took place to clarify the task.

CJ (Author): OK class. What I want you to do is…is to do this activity. Right. You first read it. And…uh. You can read it together. What we say is that when I've got mathematical…sums or things written down. Right. Either in symbols or in words…then sometimes…then it is…then it can be always true. What is always true? Like two plus 2 is equal to?...Four. That is always true. Right. Sometimes it is never true…like two minus three is equal to one…What is the answer?

Chorus answer by some girls: Negative one.

CJ: What is it?

Chorus answer by more girls: Negative one.

CJ: It's negative one. So that is never true. But sometimes we get things in between. OK. If I say that a number is always a negative…a number is a negative number. Right. A number is a negative number. Is that true, always true?

Learner: No

CJ: Why not?

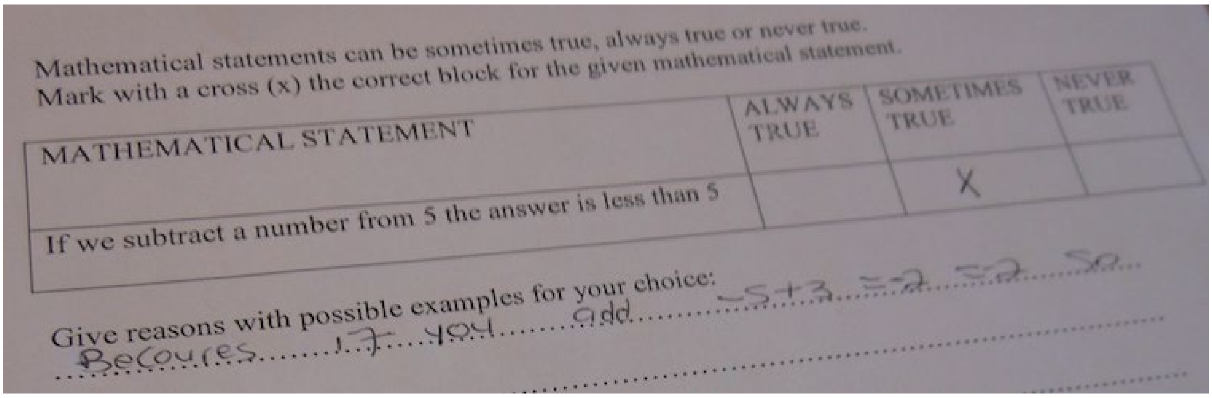
Learner A: Because…because…because…you [inaudible] alone count is also positive… negative…

CJ: OK. So the way that you count is always not negative. It can be negative but it can be other numbers also…other numbers like positive. OK. So that is a statement that is sometimes true. Do you understand it now? Do you understand the thing of always true, sometimes true, never true? OK. Now what we want you to do is there's a mathematical statement, if we subtract a number from 5 the answer is less than 5. OK. That is the statement. Now you've got to think about it, speak to your colleague, your friends about it and then you mark with a cross what you decided it is. Right. Is it always true, is it sometimes true, is it never true. And then in here [shows the section on the activity sheet] you write down what you reasons are. Right. So you can speak to one another about it and then do it. OK. So you can…You understand now what you must do?

Learner B (softly): Yes

CJ: OK. Now do it.

The learners then proceeded with the activity in groups of about four organized around those seated near to each other. They took about 11 minutes to complete the task. Some of their written responses are given in the Figures 8(a) – 8(f).

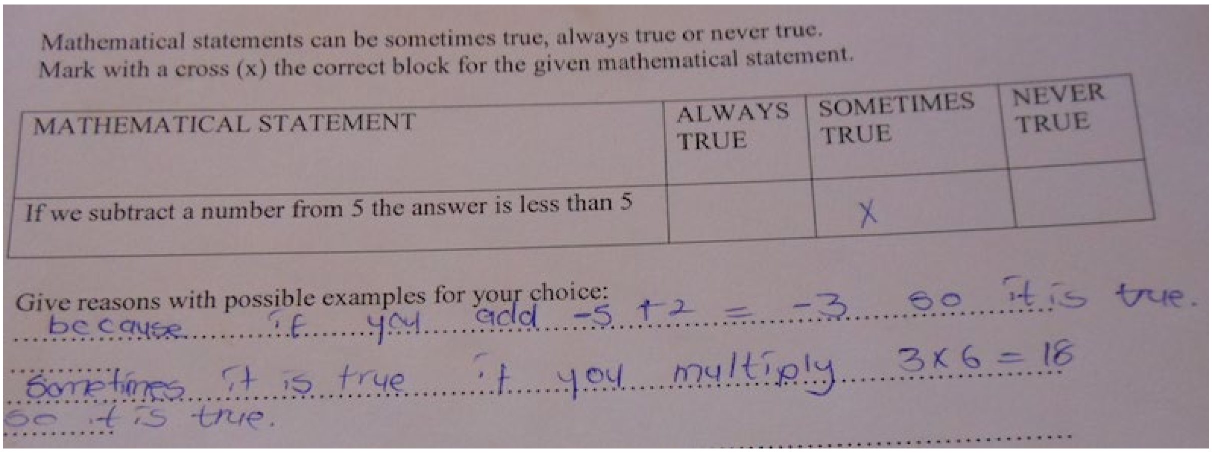


Mathematical statements can be sometimes true, always true or never true.
Mark with a cross (x) the correct block for the given mathematical statement.

MATHEMATICAL STATEMENT	ALWAYS TRUE	SOMETIMES TRUE	NEVER TRUE
If we subtract a number from 5 the answer is less than 5		X	

Give reasons with possible examples for your choice: Becoures 17 you add -5+3 = -2 =2 So...

8(a)

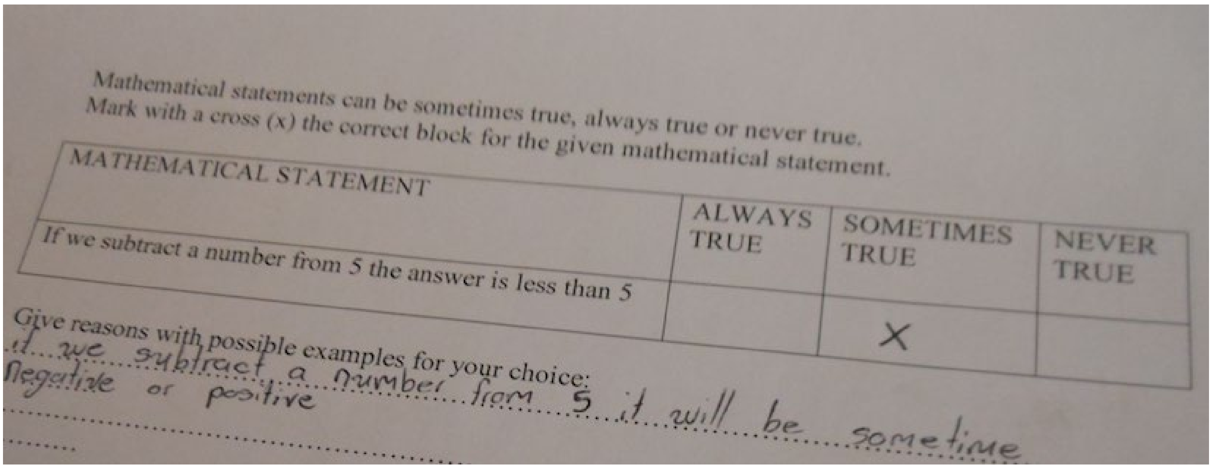


Mathematical statements can be sometimes true, always true or never true.
Mark with a cross (x) the correct block for the given mathematical statement.

MATHEMATICAL STATEMENT	ALWAYS TRUE	SOMETIMES TRUE	NEVER TRUE
If we subtract a number from 5 the answer is less than 5		X	

Give reasons with possible examples for your choice: because if you add -5 +2 = -3 so it is true. Sometimes it is true if you multiply 3x6 = 18 so it is true.

8(b)

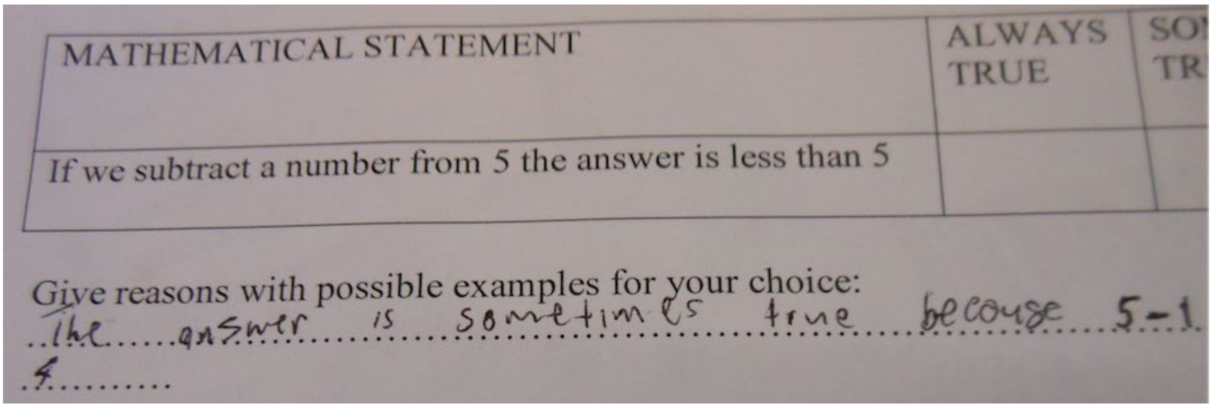


Mathematical statements can be sometimes true, always true or never true.
Mark with a cross (x) the correct block for the given mathematical statement.

MATHEMATICAL STATEMENT	ALWAYS TRUE	SOMETIMES TRUE	NEVER TRUE
If we subtract a number from 5 the answer is less than 5		X	

Give reasons with possible examples for your choice: if we subtract a number from 5 it will be sometime Negative or positive

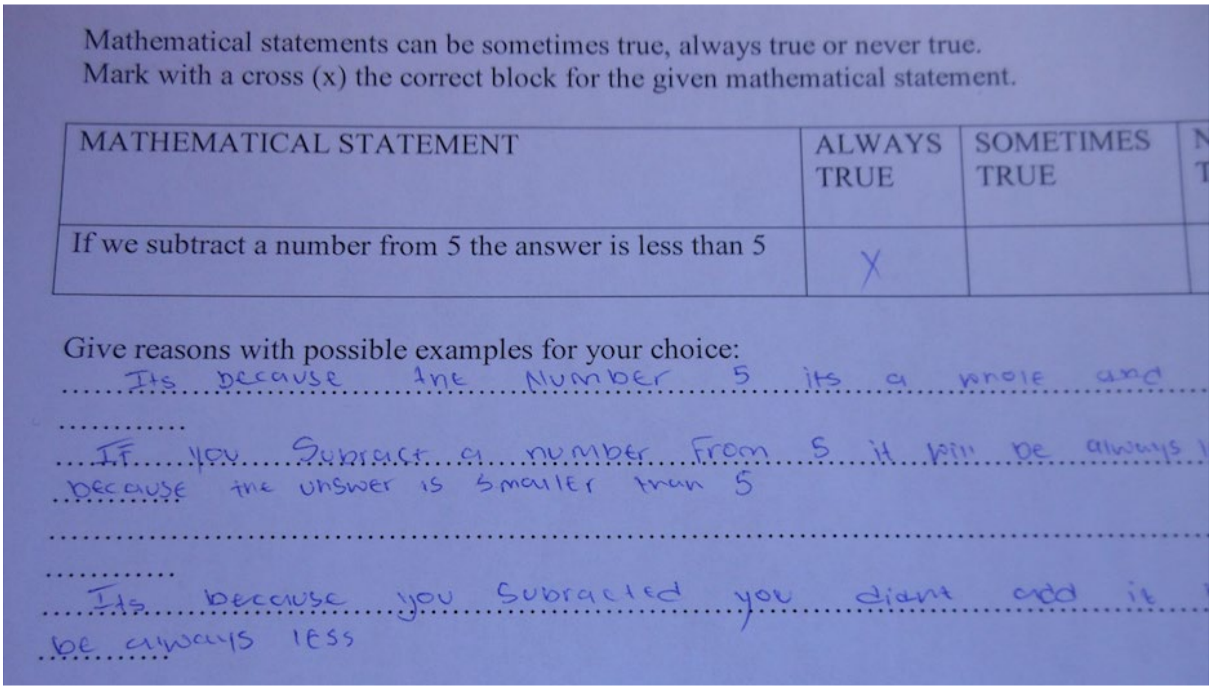
8(c)



MATHEMATICAL STATEMENT	ALWAYS TRUE	SO TR
If we subtract a number from 5 the answer is less than 5		

Give reasons with possible examples for your choice:
the answer is sometimes true becouse 5-1 4

8(d)

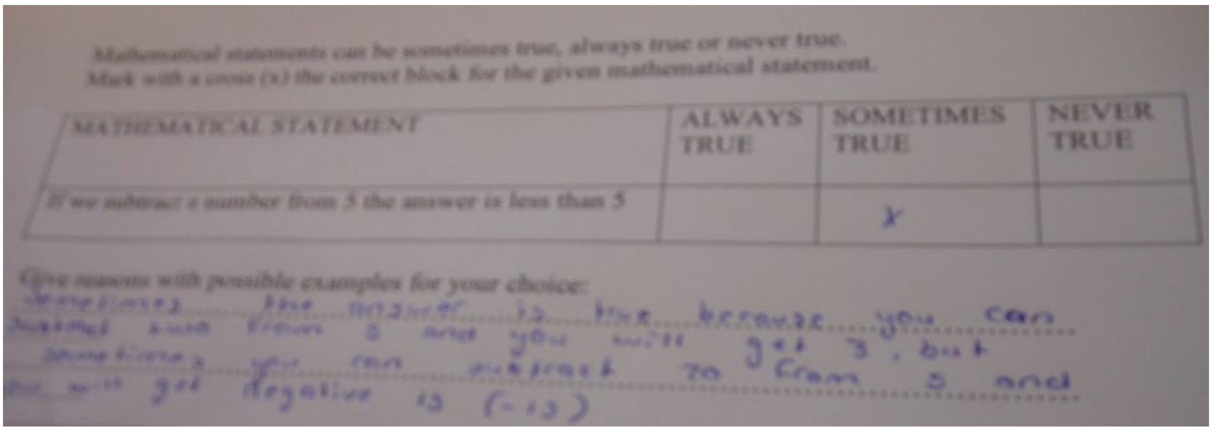


Mathematical statements can be sometimes true, always true or never true.
Mark with a cross (x) the correct block for the given mathematical statement.

MATHEMATICAL STATEMENT	ALWAYS TRUE	SOMETIMES TRUE	N T
If we subtract a number from 5 the answer is less than 5	X		

Give reasons with possible examples for your choice:
Its because the Number 5 its a whole and
If you Subract a number from 5 it will be always
because the unswer is smaller than 5
Its because you Subracted you didnt add it
be always less

8(e)



Mathematical statements can be sometimes true, always true or never true.
Mark with a cross (x) the correct block for the given mathematical statement.

MATHEMATICAL STATEMENT	ALWAYS TRUE	SOMETIMES TRUE	NEVER TRUE
If we subtract a number from 5 the answer is less than 5		x	

Give reasons with possible examples for your choice:
sometimes the answer is true because you can subtract two from 5 and you will get 3, but sometimes you can subtract 20 from 5 and you will get negative 15 (-15)

8(f)

Figure 8(a) – 8(f): Selected responses to ASN task

The above collection of responses shows that learners did not consider the subtraction of zero or a negative number from a positive number.

My reflection was in the direction on whether the designs of DMT activities of this nature should not include a pre-activity where learners' attention is drawn to the essential elements that should be focused on.

The activity was concluded by me summarizing, using an expository question-answer-approach. This summary focused on the answers of the learners, introducing the subtraction of a negative number from 5 as shown in Figure 9 and drawing learners' attention of careful reading of statements—"the answer is less than 5" in this case.

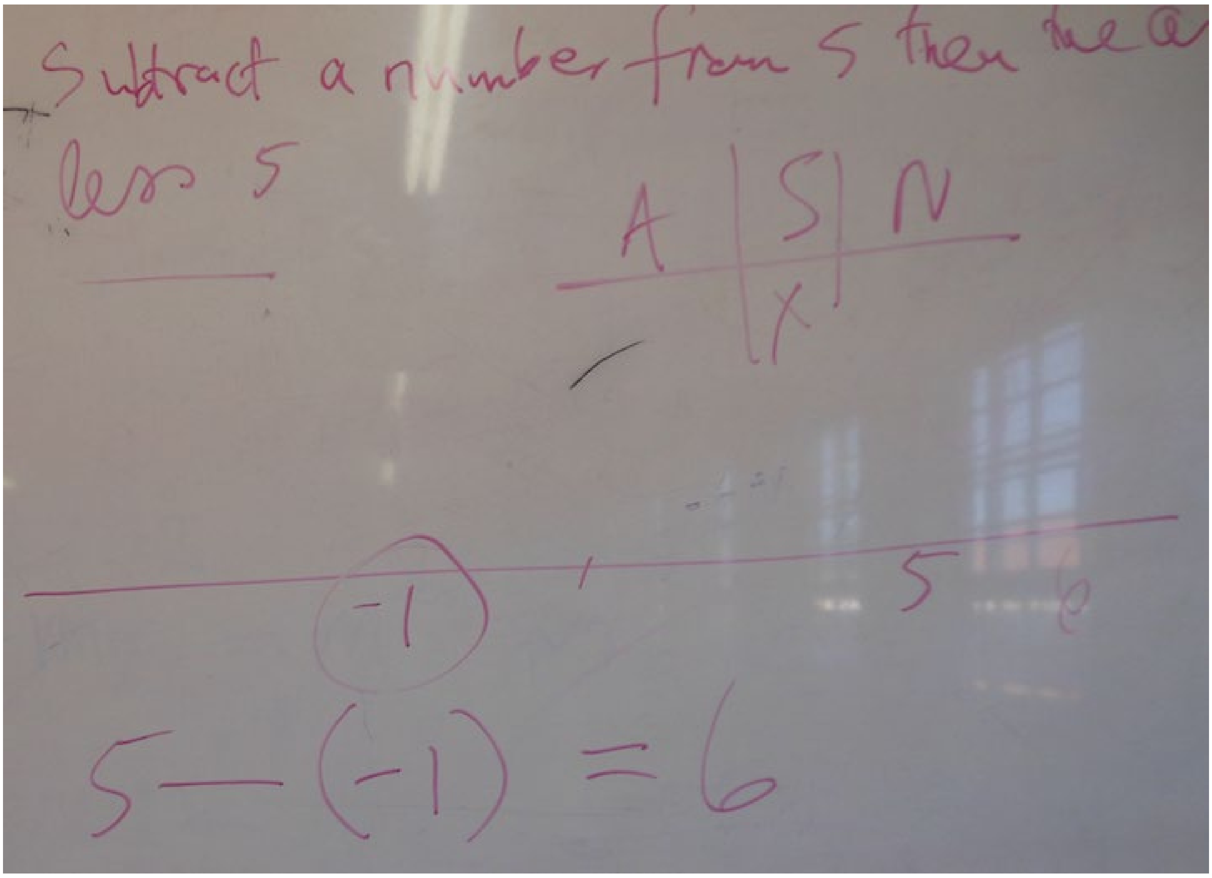


Figure 9: Whiteboard illustration of a negative number subtracted from 5

The teacher then requested the learners to continue with the work they were busy with the previous day. These were exercises written on the whiteboard as illustrated in Figure 10.

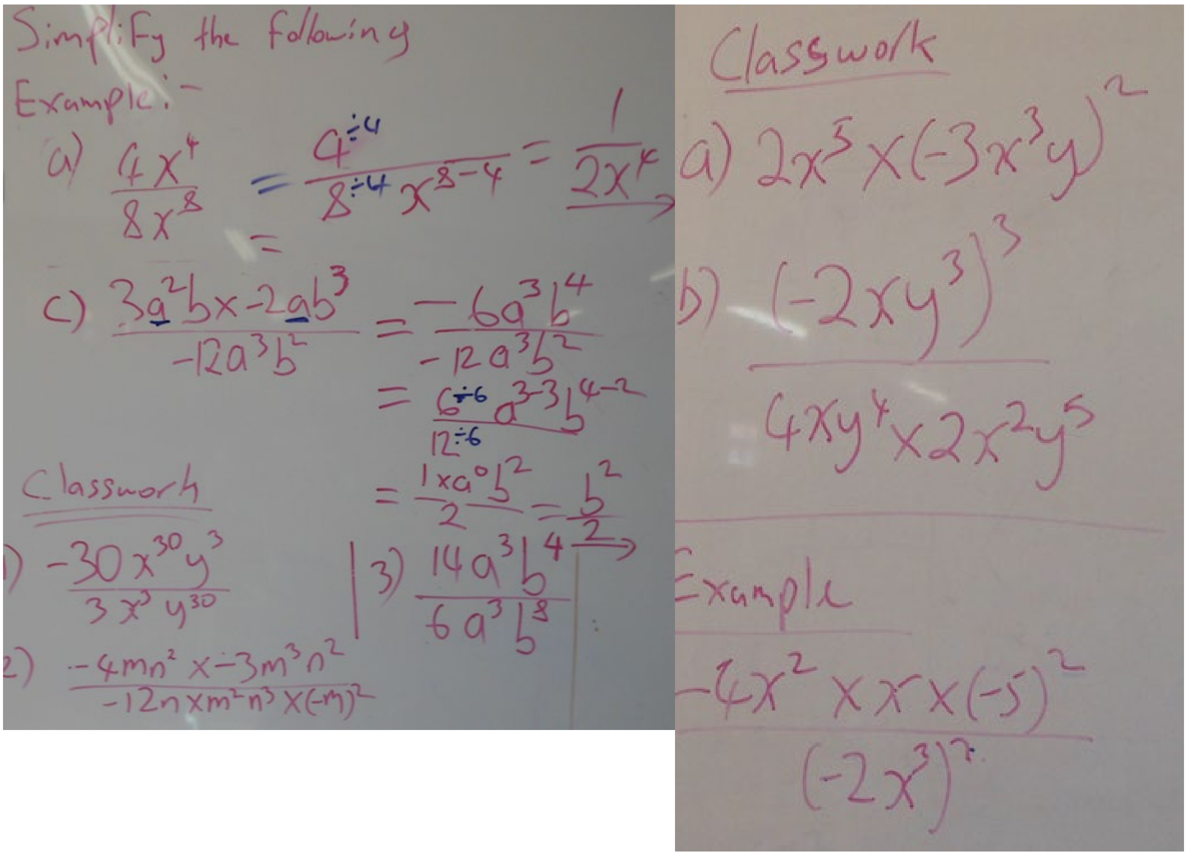


Figure 10: Work learners had to do after the ASN task

As I undertook the 175-kilometre journey back to Cape Town and reflected, I hummed "The road is long. With many a winding turn. That leads us to who knows where…" But being a Liverpool supporter I could not resist also humming "Walk on, walk on. With hope in your heart. And you'll never walk alone."

CLOSING REMARKS

I am aware that there is more description than analysis in this article. Analysis against some "theory" was not the intention. As alluded to above I was merely interested in "what do learners produce when they encounter deepening mathematical thinking tasks with me as the 'teacher' dealing with the task and their teacher afterwards, during the same period, continuing with normal teaching." Hence my story is the work produced by the learners. I return to my mentor in qualitative research, Terry Denny (who passed on in 2015), who asserts "Conclusion needers will find these few observations unsatisfactory in that they represent overriding observations" (Denny, 1977, p. 122) about what learners have produced through engagement with deepening mathematical thinking tasks but "allow readers "elbow room" to draw conclusions other than those presented directly by the writer." The story above may not be as lucid a story as Denny would have liked but hopefully, as an attempt at educational story telling it is, as Freudenthal (1991, p. 161) desires, "transmitted to others to become like their own experience."

ACKNOWLEDGEMENT

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FOUNDATION PHASE LEARNERS' KNOWLEDGE OF MEASUREMENTS: "MASS OF OBJECTS"

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Central University of Technology University of the Western Cape

Measurement of mass is a fundamental topic across the Foundation mathematics as well as it is likely to be present in most students' lives at home and in the wider world. In South Africa, a variety of systemic evaluations have shown that learners do not have the expected skills in measurement of mass. This is because of the abstract nature of concept of mass, learners find it difficult to understand. In order to construct their own understanding of mass, learners usually refer to their personal experience. This paper explored the foundation phase learners' knowledge of measurement of mass. The research reported in this study used a qualitative research method. Five grade 3 learners were involved in the interviews to answer questions involving the measurement of mass. In this exploration, learner's representations in the form of narratives were analyzed. Results show that most learners have basic understandings of measurement of mass. Moreover, results show that there are common misconceptions formed out of learners' real-life experiences. This paper recommends that teachers adapt the narrative and the drawing activities described in this study to classroom practice, this allows teachers to recognize and extend the understandings about measurement of mass which learners possess. This will help learners to progress to formal engagements with measurement of mass.
Keywords: Measurement of Mass, Learners' knowledge, Foundation Phase.

INTRODUCTION

Mass is the "amount of matter in an object, and it cannot be seen" (NSW Department of Education and Training Professional Support and Curriculum Directorate [NWS DET PS & CD], 2003). But learners may be able to feel the difference in mass /weight) physically in terms of heavy, heavier, heaviest etc. Therefore, of the mass' abstract nature, learners often find the concept of mass difficult to understand (Gifford, 2005). In this study, understanding the concept of mass refers to using it for their own purposes, talking about their measurement ideas, representing measurement processes in ways which make sense to them, and becoming more aware of their own measurement thinking (Whitebread, 2005). In South Africa, Foundation Phase Mathematics forges the link between the child's pre-school and outside school life on the one hand, and the abstract Mathematics of the later grades on the other hand. In the early Grades learners should be exposed to mathematical experiences that give them many opportunities “to do, talk and record” their mathematical thinking (Department of Basic Education [DBE], 2012,8). This is the case with the measurement of mass in the Foundation Phase. According to Gifford (2005), in order to construct their own understanding of mass, learners usually refer to their own experiences. These experiences vary from learner to learner, reflecting their different home environments, family interests, and personal circumstances.

Understanding of measurement

The current study is restricted to a framework based on measurement learning frameworks of Clements and Stephan (2004); Piaget, Inhelder, and Szeminska (1960); and Board of Studies NSW (2002). There are many developmental sequences for measurement learning presented in the literature. Three examples sequences of measurement learning are that of Clements and Stephan (2004); Piaget, Inhelder, and Szeminska (1960); and Board of Studies NSW (2002) (see Table 1).

This framework is divided into two levels namely: the emergent measurement and the proficient measurement. According to Whitebread (2005), emergent measurement encourages children to develop an understanding of measurement by using it for their own purposes, talking about their measurement ideas, representing measurement processes in ways which make sense to them, and becoming more aware of their own measurement thinking.

Table 1 Measurement learning frameworks of Clements and Stephan (2004); Piaget, Inhelder, and Szeminska (1960); and Board of Studies NSW (2002).

Clements and Stephan (2004)	Piaget, Inhelder, and Szeminska (1960)	Board of Studies NSW (2002)
Awareness of continuous attributes, but unable to quantify or measure accurately Use of words that represent quantity of an attribute, direct comparison, and recognition of equality or inequality Connect number to quantity, identify unit of measure, and measure through unit iteration.	Not being capable of measurement construction of units is impossible. Ability to use a common measure, use of unit iteration Direct measurement is possible	Identifying the attribute and comparison Informal units Formal units Applications and generalizations

By comparison, proficient measurement requires: comprehension of measurement concepts, operations and relations; skills in carrying out procedures flexibly, accurately, efficiently and appropriately; ability to formulate, represent and solve problems; and capacity for logical thought, reflection, explanation and justification (Kilpatrick, Swafford & Findell, 2001). This study explores Foundation Phase children's knowledge about the measurements of mass.

Understanding the measurement of Mass

Naturally, learners learn to measure mass using informal units before progressing to the use of formal units (Macdonald, 2011). These suggested learning pathways match the sequencing of content around measurement of mass in the Curriculum and Assessment Policy Statement (DBE, 2012), where they learn to measure using informal units in grade R, grade 1 and at the beginning of grade 2, before using formal measurement in grade 2 and 3. Every experience provides opportunities for learners to explore mass in a familiar environment, discovering its properties, and thus constructing their own knowledge (Charlesworth, 2005). Learners can communicate their understanding of mass by talking about their own experiences. Individuals may use narratives for meaning making in addition to using them for sharing their experiences in stories (McAdams, 1993; Reissman, 1993). Learners may use narratives to tell about one object is heavier/lighter than the other one, to tell about a big box/small may look heavier that the small box/big box. But it is not the case. Furthermore, learners may tell about the mass of objects floating and sinking in the bath, in a swimming pool or at the beach. Another way learners can communicate their understandings about mass is by drawing and providing explanations of their drawings. This process is termed "drawing-telling" (Wright, 2007) and its captures the ways in which learners make sense of the concepts of mass and identifies the prior experiences and background knowledge brought to the concept by learners (Woleck, 2001).

RESEARCH QUESTIONS

Guided by the literature review the following research question was raised: What knowledge of measurement of mass do foundation phase learners have at the time they start their grade 3?

METHOD

Research design

The research reported in this study used a qualitative research method. The theoretical perspective of this investigation is constructivism. Hatch (2002, p.15) addressed the quest of a constructivist researcher as "individual constructions of reality compose the knowledge of interest to constructivist researcher". For this research, in order to study learners' understanding of measurement of mass, the researcher listened to learners about their experiences of learning of measurements and their knowledge of mass.

Research Instrument

The data were collected from a primary school in the Free State, South Africa. The learners had just commenced their grade 3 in January. They started after having completed grade 2 successfully. Data collection commenced at the beginning of February 2019, when the learner had been at school for roughly three weeks. The data collection method included individual interviews consisting of five questions. The main purpose of these interviews was to study learners' understanding of the measurement of mass. The 30-60 minutes long interviews were conducted after school. The narrative interview protocol, which was designed to be semi-structured and open-ended, was used. For this study, the suggested narrative interview probes are "Tell me about…" (Reissman, 1993, 2000). Another important feature of narrative interviews in this study is that the researcher accepts the leading role of the participant because the participant is the knowledge holder (Bruner, 1990; Reissman, 2000).

Participants

Participants for this study were drawn from the foundation phase at a primary school in the Magaung Municipality, Free State Province, South Africa. The researcher conducted these interviews with 5 grade 3 learners selected by means of a purposive sampling strategy. The researcher selected learners who were willing to be part of this research. To capture tone and allow greater authenticity, transcriptions were precise and included utterances such as "aah" and "yeah." The researcher verified the accuracy of each transcription

Reliability and Dependability

Several measures were taken to enhance the credibility and dependability of data. In this study, triangulation involving the use of different methods, specifically semi-structured and open-ended took place in order to promote confidence that the researcher accurately recorded the phenomena under scrutiny. Peers and academics were given the opportunity to scrutinise the research. Previous research findings were examined in order to assess the degree to which the research's results are congruent with those of past studies. In order to address the dependability issue more directly, the processes within the study should be reported in detail, thereby enabling future researchers to repeat the work, if not necessarily to gain the same results. Thus, the research design may be viewed as a "prototype model" (Shenton, 2003;71).

RESULTS AND ANALYSIS

In this section we will present the results and discussion according to the research question that was raised for this study. Data analysis was accomplished through qualitative analysis of learners' responses to the interview. The data analysis was focused on participants' experiences of measurements of mass. The interview respondents are referred to as participant 01, participant 02, participant 03, participant 04 and participant 05.

Analysis of learners' responses to close-ended questions

Questions 1.1, 2.1, 3.1 and 4.1 were close ended ones and these questions aimed at capturing responses on how learners compare the mass of objects by observation using descriptors such as heavy or light. In these instances, learners were asked to observe or pick objects and state which one is heavy or light. For each of the question 1.1; 2.1; 3.1; and 4.1, themes emerged from our analysis of the responses and the results are presented as follows:

In ***question 1.1***, participants were asked to observe an A4 page and a cell phone, and to tell the researcher which one of the two objects was heavy, or which one was light?

In terms of ***heavy***, this is what participants had to say:
Participant *03 and participant 05*: *The cell phone is heavier than the paper*.

In terms of ***light (in contrast to heavy)***, participants shared the following:
Participant *01, 02 and 04*: *The paper is lighter than the cell phone*.

In ***question 2.1***, participants were asked to look at the two boxes on the table and tell the researcher which one of the two boxes was heavy?

In terms of ***heavy***, all the five participants had to say: I see the big box look heavy…

In ***question 3.1***, participants were asked to tell the researcher**,** between the boxes they have picked up which one is heavy.

In terms of ***heavy***, all five participants had to tell: the small box is heavier than the small paper.

In ***question 4.1***, participants were asked to look at a leaf and a piece of brick on the table and tell the researcher which was the heavier or lighter.

In terms of ***heavy***, here is what participants had to say:
Participant *01 and participant 03*: *I think the piece of brick is heavy*.

In terms of ***light***, here is what participants had to tell:
Participant *02, 04 and 05*: *The leaf is lighter than the piece of brick*.

Analysis of learners' responses to open-ended questions

Questions 1.2, 2.2, 3,2 and 42 were open ended ones and participants were asked to tell the researcher why one object was heavier or lighter that the other. From the responses to these open-ended questions the researcher aimed at capturing responses on how participant understood the concepts of objects mass. For each of these questions themes emerged from our analysis of the responses and the results are presented according to those themes as follows:

In ***question 1.2***, participants were asked to tell the researcher why one of the objects was lighter than the other one.
In terms of ***flat/thin***, participants had the following to say:
Participant *01*: *The paper is lighter because the paper is flat*.
Participant *02: If I look at the two objects the paper is thin. It is why its light*.

Participant *03: I think the paper his light because the paper is thine.*
Participant *04*: *I can see that the paper is light because it is flat*.

In terms of ***things inside***, the participants had to say:
Participant *01: If I open the cellphone, I find many small things inside.*
Participant *04: I know that if you open the cell phone you have small things in side.*
Participant *05: I break my cell phone and I see many things are in side. The paper is only one paper.*

In terms of ***big***, here is what participants had to tell:
Participant *02: I think the cell phone is big. The wind can take the paper not the cell phone*.
Participant *03: I see the cell phone is big than the paper.*
Participant *05: Paper is light because, I think that the cell phone is big. I know the paper is always light.*

In ***question 2.2***, After having chosen the box that was heavy without lifting the boxes, participants were asked to tell the research why one of the boxes was heavy.

The first them is ***big/large***, here is what the participants had to express:
Participant 01: *when I look at the box the big box is heavy than the small box.*
Participant 05: *I know that the big box is heavy because it is big.*

The second theme is ***many things inside***, here is what the participants had to tell:
Participant 02: *the big box is heavy. I know we can put many things inside the big box.*
Participant 03: *the big box has may thing inside*.
Participant 04: *the big box is heavy. My mother buys a fridge in a big box. It was heavy.*

In ***question 3.2***, After picking up the two boxes, participants were asked to tell the researcher why one of the two boxes is heavier than the other,

In terms of ***many things insides***, here is what the teachers had to say:
Participant 01: *The small box has many things inside it. But the big box nothing.*
Participant 03: *I can see that there are many things inside the small box.*
Participant 02: *I cannot lift the small box many things are inside.*

In terms of **heavy *things insides***, here is what the teachers had to say:
Participant 02: I think heavy things are inside the small box.
Participant 03: *The things in the small box are heavy.*
Participant 04: *I think too much that something heavy like a brick is in the small box.*

In ***question 4.2***, participants were asked to trough the two objects in the water, tell the researcher which one floated and why the objects floated?

In terms of ***leaf that floated, it is light***, here is what the participants had to say:
Participant 01: *A leaf floated because it light than a piece of brick.*
Participant 04: *A brick floated because it is lighter*.
Participant 05: *The leaf floated because it is lighter that the piece of brick.*

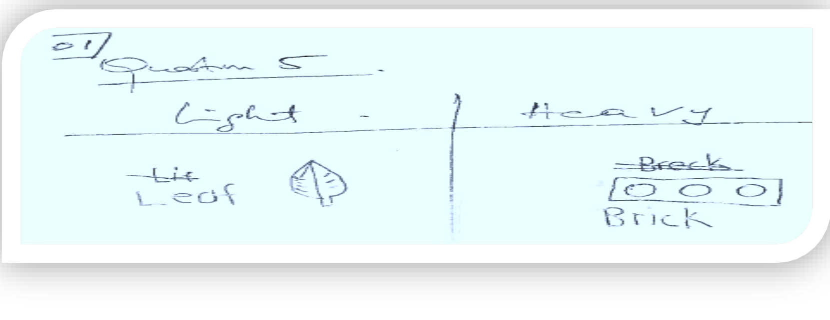
In terms of ***leaf floated, it is thine/flat***, here is what the participants had to say:
Participant 02: *A leaf floated because it thine.*
Participant 03: *A leaf is floated because it flat.*

In terms of ***leaf sinks***, here is what the participants had to say:
Participant 04: *But sometime the leaf sinks. I see it in our swimming pool.*
Participant 01: *But in our swimming pool there are more leaves under the water during winter.*

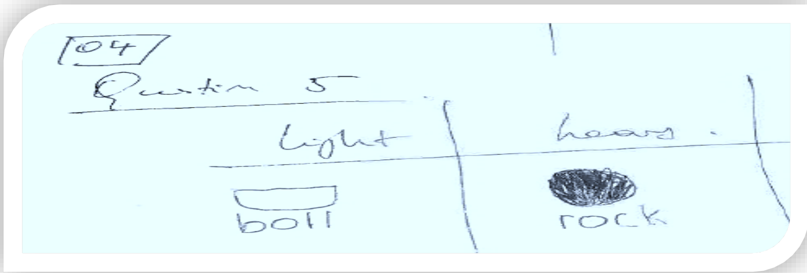
In terms of ***not float, it is heavy***, here is what the participants had to say:
Participant 02: *A piece of brick cannot float if it is small. I do not see a brick floating even the sand.*
Participant 03: *If the brick was flat it can float*
Participant 05: *I think if the object is lighter it always floats, and a piece of brick is sinking always because it is heavy.*

Furthermore, in ***question 5.2*** participants were asked to draw two objects of their choices. One must be heavy and one light, and tell how they can find which one is heavy or light without using a scale?

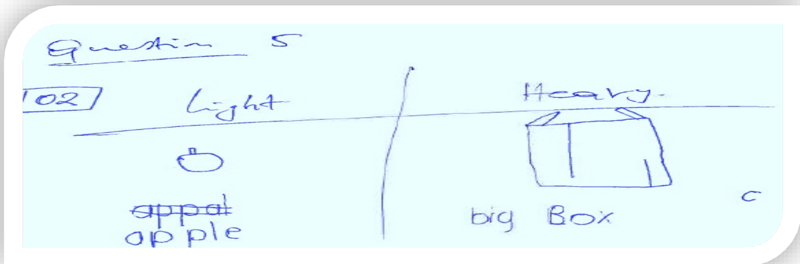
In terms of ***light/heavy***, here is what the participants had to tell about their drawings:
Participant 01: *I draw a leaf and a ball. The leaf is light than the brick.*



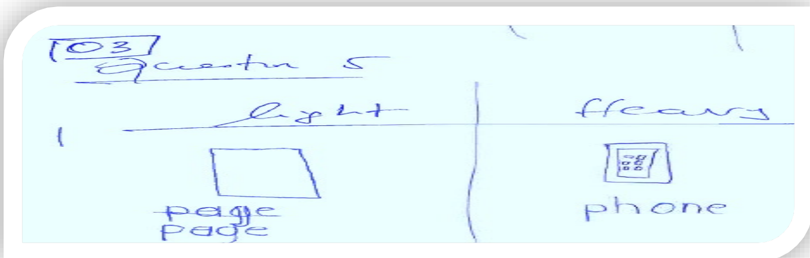
Participant 04: *My drawing represents the ball and the rock. The ball is light than the rock.*



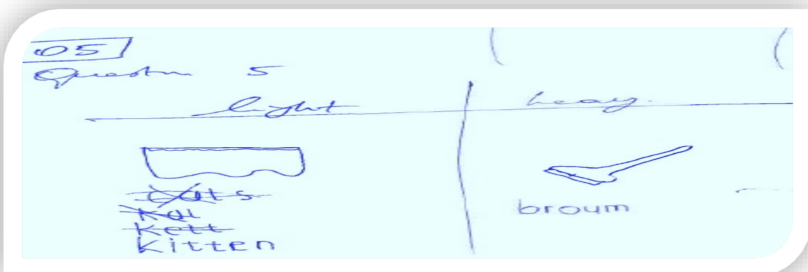
Teacher 02: *In my drawing an apple is lighter than the big box.*



Participant 03: *The drawing shows a page and phone. The phone is heavy than the page.*



Participant 05: In my drawing broom is heavy than a kitten.



In terms ***thinking***, here is what the participants had to tell about their choices:

Participant 01: *I can just see, in my mind the leaf is light, and the brick is heavy. It is always like that the leaf is light than the brick.*

Participant 03: *I just think about a page is light and phone is heavy. I do not need to lift the object. I just guess.*

In terms of ***lifting***, here is what the participants had to say about their drawings:
Participant 02*: I need to lift the box and the boll to tell which one is light or heavy.*
Participant 05: *I will lift the kitten and the broom; small kitten is light. Big kitten is heavy than the broom.*

In terms of ***holding in hands***, here is what the participants had to tell about their drawings
Participant 04: *Sometime I can hold the apple and the box in my hands if the two objects are small. I can lift if the box is big.*

DISCUSSION

With reference to the research question on foundation phase learners' knowledge of measurement of mass one of the results was that most of participants were able to compare objects mass using the descriptors as either "heavy" or "light" and give raisons for their choice. As evidenced in their responses, participants demonstrate a basic understanding of comparison and used of descriptors (heavy or light). For examples in terms of light, Participant 01, 02 and 04 said: *"The paper is light than the cell phone"*. In terms of heavy, Participant *01 and* participant 03 revealed: *"I think the piece of brick is heavy"*. Furthermore, participants demonstrated that their responses were based on or defined by their experiences in identifying objects (heavy or light), in their mind (thinking), by lifting, or handling. For example, Participant 03: *"I think the piece of brick is heavy"*. Participant *02: "if I look the two objects the paper is thin. It is why its light*". Similar results were observed by Macdonald (2010, p. 4) who indicated that through personal, naturalistic engagements with measurement, children came to learn first hands the characteristics of mass.

To describe their drawings, in their narrative participants revealed that they either identify objects (heavy or light) in their mind (thinking), by lifting, or holding objects in their hands. For example, Participant 01, 02, 03, 04 and 05 drawings were a representation of their personal experiences. In terms of describing drawings in their mind (thinking), Participant 01 revealed: *"I can just see, in my mind the leaf is light, and the brick is heavy. It is always like that the leaf is light than the brick"*. In terms of lifting objects, Participant 02 said: *"I need to lift the box and the boll to tell which one is light or heavy*". By lifting they can fill which one is heavy or light. Some participants were able to represent themselves comparing the masses of items by holding them in their hands. A process known as “hefting”. Participant 04 is one of them. Participant explained: *“Sometime I can hold the apple and the box in my hands if the two objects are small. I can lift if the box is big”.* Tucker, Boggan and Harper (2010) find that, children experience measurement concepts personally, through stories that relate the use of measurement to their daily lives and through hands-on activities.

However, a common misconception is that larger things weigh more (Gifford, 2015). Most participants’ responses showed this misconception. For example, Participant 05: *“I know that the big box is heavy because it is big”.* Participant offer the similar argumentation: *“Paper is light because, I think that the cell phone is big. I know the paper is always light”.* the mass of an object may not relate to its volume. Therefore, teaching about mass is to distinguish it from volume (Booker, Bond, Briggs, & Davey, 1997). This indicates that object size can influence perceived heaviness, even when differently sized objects contain the same volume of material (Plaisier & Smeets, 2015). Learners need to be given opportunities to experience and discuss large light objects and small heavy things. Furthermore, participants classified objects as either heavy or light based on whether things could be inside the object. For example, Participant 01, said: *“The small box has many things inside it. But the big box nothing”.* Participant 02 also based his narrative of mass on his experiences of heavy things being those which have other things inside them: *“I think heavy things are inside the small box”. In their research,* Plaisier and Smeet’s (2015) research, showed that heaviness perception is influenced by our expectations, and larger objects are expected to be heavier than smaller ones because they contain more material.

Additionally, Participant 01 incorporated notions of theory 2 in is explanation that is, that “big” equates to “heavy”. For the participants the more things can fit inside an object, the heavier it must be. Additionally, participants explained that if an object float, it is light. Participant 05 said: *“The leaf floated because it is lighter that the piece of brick”.* This classification is based on the property of density, rather than mass. According to Kohn (1993), the learner’s confusion might be justified and profound because what it means for some things to be of high in density is sometimes explained to learners as being “made of heavy stuff. To overcome these misconceptions, learners need practical examples which distinguish mass from density. Activities which explore objects of the same size but that have different mass.

These personalised understandings provide a useful starting point when teaching measurement of mass. Already the learners show some knowledge about mass and can progress to formal engagements with mass measurement.

CONCLUSION

Findings show that learners have some understandings of measurement of mass. Learners have individualized, distinctive ways of understanding the measurement of mass. By allowing them to come with a narrative and representation of their understanding in meaningful ways, we can identify the prior experiences with the concept which they have. Moreover, we can identify at an early stage any misconceptions which may be held. These misconceptions have value because they are formed out of learners' real-life experiences, and these experiences provide the most meaningful starting points for measurement learning. The narrative and the drawing activities described in this study could easily be adapted to classroom practice, and such adaptation(s) would allow teachers to both recognize and extend the understandings about measurement of mass which learners possess.

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STRIVING FOR RIGOUR IN A COMPULSORY MATHEMATICS COURSE FOR TEACHER STUDENTS

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Are teacher educators fulfilling their roles to train future teachers to be more knowledgeable others to their learners? The author presumed that engaging teacher students with high mathematical demand tasks in relation to their misconceptions prepares them for that. This paper fathoms the learning difficulties teacher students encountered in a mathematics content course at an ITE institution in Johannesburg, South Africa. It examines how issues of depth and rigour are considered in the course and explores how successful deep mathematics learning occurred in 2014. The sample consisted of 653 first year students who took the course. At registration, these students had variable post school mathematical attainment. Students' test marks in the course for semester 1 and 2 were analyzed and trends in performance noted. Interviews were done with some students to determine the difficulties they had with some problematic but elementary mathematics concepts. Interventions were done in the form of weekly tutorials geared to address students' difficulties and deepen mathematical learning. Comparison of pre and post intervention performance showed that engagement of students' errors in high demand mathematics tasks, and keeping the mathematical demand high lowers performance gaps that earlier appeared at the beginning of the course.

INTRODUCTION

In recent times, academic rigour has become a topical issue in South African education. For instance, a special issue (SI) of the journal Perspectives in Education, 34(1) in 2016 was dedicated to the topic 'Academic Rigour and Depth in initial Teacher Education Institutes (ITEs)'. Contributors to the SI for example Taylor (2016) argued that ITEs need to provide strong theoretical bases constantly interrogated by practice. Academic rigour requires a curriculum design, implementation and assessment that takes into account all levels of Bloom's (1956) taxonomy namely; knowledge, comprehension, application and critical thinking. For Rembach and Dison (2016), academic rigour occurs through the SOLO taxonomy (Structure of Observed Learning Outcomes) (Biggs & Collis, 2014). In the final analysis, academic rigour is linked to a deeper, rather than surface understanding of a discipline in which students can apply acquired knowledge in solving challenging problems. This calls for the ability to analyze, critically evaluate and synthesize information. Students exposed to a curriculum that has academic rigour have agency; are self-regulated and self-directed (Makonye, 2016). The argument is that learners struggle to meet curricula demands because their learning is superficial rather than deep. For Stein, Smith, Henningsen and Silver (2000), if learners are exposed only to a curriculum with low mathematical demand tasks of memorization and procedures without connections; that is a curriculum with no rigour, they cannot be expected to solve problems requiring 'doing mathematics' in which there are no predetermined solution paths.

Lincoln (2010) links academic rigour to high standards and expectations in education in which there is more emphasis in process more than the product. Academic rigour is seen as concerned with critical thinking.

This study focusses on how the issues of academic depth and rigour in a compulsory mathematics course for student teachers (herein called Mathematical Routes at our institution) were addressed when all the students do not major in mathematics. At the Initial Teacher Education (ITE) institution that this research was done, Mathematical Routes (MR) is a yearlong mathematics course studied by all pre-service student teachers. Mindful of the fact that some of our students register for the ITE course with unproven mathematical attainments, the focus of MR is developing quality conceptual (and also procedural) understanding (Hiebert and Lefevre, 1986) of basic mathematics. Students' lack of appropriate mathematical knowledge could be a sign that they see mathematics as composed of unconnected and meaningless concepts and procedures. The researcher regards it as taken that all teachers whether they be foundation phase or not inevitably deal with mathematical concepts in one way or another at their stations – even if they do not directly teach mathematics per se. The researcher argues that for this reason all prospective teachers need to develop a sound understanding of basic mathematics (Fehr, 2010), and very importantly also, to understand the subject deeply.

Makonye (2009) has argued that some students who come to train as teachers had little success with high school mathematics. To some extent, this lack of mathematical proficiency (Kilpatrick et al., 2001) is not wholly a fault of their own because as Mkhwanazi, Brijlal and Bansilal (2014) have found out - when high school mathematics teachers where assessed on previous Matric examination tasks, the competency of most of them was below expectation. In their study, these researchers investigated teachers' knowledge of the mathematics they were currently teaching. Data was generated from in-service teachers (n = 253) studying toward the Advanced Certificate in Education at a university in South Africa. The mean mark for the performance was 57%. Also noted was that the participants scored poorly on problem solving tasks. The authors doubted these teachers' ability to competently teach mathematics at high school. I support the authors' stance because common content knowledge (CCK) (Ball, Hill and Bass, 2005) is the basis of specialized content knowledge (SCK) (Ball, Hill and Bass, 2005) which is key for one to begin to teach mathematics with expertise. Therefore some students come from disadvantaged schools where they are taught with teachers with insufficient mathematics content knowledge.

It is important for teacher educators to notice; hear and listen to their students so that they can be able to engage with their students' mathematical lag. Noticing, hearing and listening to students is formative for teaching (Davis, 1997; Bauersfeld, 1988). When teachers appropriately diagnose students' levels of mathematical competency, they can engage them via mathematical models and representations that can help them to learn better. Use of different representations helps students to realize that mathematics makes sense (Makonye, 2017). Also, observing mathematics at work particularly by way of problem solving has the power of convincing students of the usefulness of mathematics (van de Walle, 2003). Indeed students should be exposed to truly problematic tasks so that mathematical sense making is practiced and consolidated (Marcus and Fey, 2003; DBE, 2011).

Also the MR course aims to inculcate in students how the mathematics they learn may also be best taught so that school children can sensibly learn it. In effect the course aims for students not only to acquire fundamental conceptual and procedural knowledge of mathematics (Hiebert and Lefevre, 1986) or relational and instrumental understanding of mathematics (Skemp, 1987), but aims for students to develop Pedagogical Content Knowledge (PCK) (Shulman, 1986; Ball, Hill and Bass, 2005). According to Shulman (1986), PCK involves "the ways of representing and formulating the subject that makes it comprehensible to others" (p. 9). Although PCK is not a major thrust for this course, The researcher takes every opportunity to discuss ways the mathematics being learnt can be best taught to school children so that they can meaningfully understand it.

RESEARCH PROBLEM

Sosibo et al. (2016) argue the 'Initial teacher education (ITE) programmes are expected to prepare teachers who have the capacity to develop conceptually strong, responsive and inclusive teaching practices' (p.1).The gross underperformance of South African learners in mathematics (and science) in international achievement tests prompted the government to introduce national annual achievement tests. This underperformance still persists (see Trends in Mathematics and Science Study (TIMSS); Annual National and Assessments (ANA)) and is well documented (see for example Howie 2001, 2004; Moloi and Chetty 2011; Taylor and Taylor, 2013). In the Annual and National Assessment tests (ANA), Grade 9 Mathematics national average scores were 13% in 2012, 14% in 2013, and was even lower at 12% in 2014. These low results account for students avoiding the 'pure" mathematics option at grade 10 and settling for Mathematical Literacy. Such choices do not augur well for the South African nation which greatly needs a highly trained workforce to power the economy.

Earlier, researching on the quality of mathematics teachers, Modiba (2011) and Carnoy, Chilisa and Chisholm (2012) argued that even the best teachers need adequate subject matter knowledge and that the 'low mathematics achievement trap' (p.43) in South Africa was caused by teachers' suspect MCK in the mathematics they teach.

The above research and others (Adler, 1995; Essien, 2010), show that the work of teacher educators is clear cut, for it is the ITE institutions that are at the centre of this crisis. Researchers and practitioners need to act on this; at least with respect to new teachers who graduate from ITE institution. They need to produce teachers who at the very least possess adequate MCK so that they will more knowledgeable others to their pupils. As Vygotsky (1978, 1986) argued, learning occurs in a learner's Zone of Proximal Development (ZPD) when learning is mediated and scaffolded by a more knowledgeable other (MKO). The researcher also argues that the best teachers are the more knowledgeable others in mathematics content. One cannot develop pedagogy if one does not have the content. Teachers with adequate MCK often will have rigorously studied mathematics at school and at ITEs. Although MCK cannot explain everything, such teachers tend to be better mathematics teachers than those with weak MCK (Ball, Hill and Bass, 2005).

Against the aims of Mathematical Routes discussed above, the researcher states that the present intake of our ITE students have varied school mathematics qualifications. There are three groups, those who studied "pure" Mathematics, those who studied Mathematical Literacy and those students who did not do mathematics all. In the third group are some students who left school decades ago (see Daphne in the data analysis); and without any mathematics credentials. Even some of our students who left school recently, dropped mathematics or mathematical literacy at the FET phase saying that these subjects were too difficult for them. Yet all these students, despite their unequal school mathematics attainment at registration are expected to study the same Mathematics Routes curriculum here. They are also expected to teach it to young school learners if they specialize at Foundation Phase.

AIMS OF THE STUDY

This study explores the teaching of a compulsory mathematics course to first year students with the aim to ensure depth and rigour in a background where on one end some students have very shaky mathematics qualifications and on the other end there are students who have very high mathematical competency. In the foreground is this researcher's concern that academic rigour and depth is critical in this basic course so that the demands of the subject are not compromised in a way, so that when students eventually graduate they will have a strong mathematics background to hold on to which will help them to adequately support learners to learn mathematics. Only then can one hope for the learner mathematical achievement to improve, when the teachers who teach them, at least possess adequate MCK and are the MKO.

RESEARCH QUESTIONS

1. With respect to the teaching and learning of the 2014 MR course at our ITE institution,
2. What are the various students' misconceptions on the Foundation Mathematics Routes Course? and;
3. What factors affect the academic depth and rigour in the Foundation Mathematics Course given that students have variable mathematics competences when they start the course?
4. What shifts occur if any, in academic rigour and depth, when students are engaged with high mathematics demand tasks in view of the errors and misconceptions they have on the mathematics?

CONCEPTUAL FRAMEWORK

To ensure depth and rigour in ITE institutions, the curriculum can be critiqued through the lens of Bernstein (2000). Bernstein's lens analyze curriculum as teacher controlled versus student controlled; and whether the curriculum is specialized or generalized. Bernstein used the terms 'framing' and 'classification' to analyze the curriculum. According to Bernstein (2000), framing of the curriculum relates to the control that the teacher or learners have during the learning process; while classification relates to the extent that curricula subjects, for example mathematics or biology stand distinct from other subjects or are integrated with them. Weak classification implies that the curriculum avoids over-specialization and curricula subjects tend to overlap, to integrate; the boundaries between different subject disciplines are indistinct and collapsed. In strong classification, the subjects stand in highly specialized and isolated silos. In strong framing the teachers control every facet of the learning process in a top down approach. This means that the teaching is centred on the teacher and learners voices are of secondary importance if at all. In this paper the author argues that for student teachers to have strong MCK, it is necessary that the mathematics curricula have strong classification; and to ensure strong classification there must be strong framing, teacher educators must have strong control of how deep mathematics concepts are taught to students.

METHODOLOGY

The participants of the research were all first year students in 2014 (n=640) at our ITE institution. Two tests (see appendix for sample test tasks); one for semester 1 and one for semester 2 were analyzed for student's performance and tasks with low scores noted. Also, questionnaires were given to students for them to express how they felt about the maths routes course. Students were asked to indicate their school leaving mathematics qualifications, year of leaving school as well as the level achieved. Students were informed not to write their names and were urged to be as open as possible on what they thought about the rigour and academic depth of Maths Routes. They were asked to comment with respect to topics and concepts they found easy, difficult, important, enriching, timewasting, unnecessary as well as topics and concepts they felt needed to be included in the course to improve it.

Also, interviews were held with a few low scoring students to assess the levels of challenges they faced as they solved some maths routes tasks of interest. Students who struggled with the most basic concepts were put in separate tutorial groups to help them deal with their mathematical difficulties. Their performance at the year-end examinations in the course was noted.

While students commence the course with different levels of mathematics competency, they study together in mixed ability classes with the aim of them all developing high levels of mathematical competency by the time they graduate.

Notice the interacting factors particularly the role played by students' errors and misconceptions and the tutorials that focus on these as teaching resources for the course to be too strongly classified (Bernstein, 2000).

DATA ANALYSIS

Analysis was done quantitatively and then qualitatively.

Quantitative data analysis

The tables and bar graphs below represent the performance of students (n=650) in the Maths Routes tests in 2014.

TABLE 1: Students' performance in MR test 1 and test 2

Mark	0-49%	50-59%	60-69%	70-79%	80-100%
Percentage in each class:					
Test1 March, 2014	36.7	21.0	20.2	5.5	16.6
Test 2, October, 2014	20.2	24.1	27.1	8	20.7

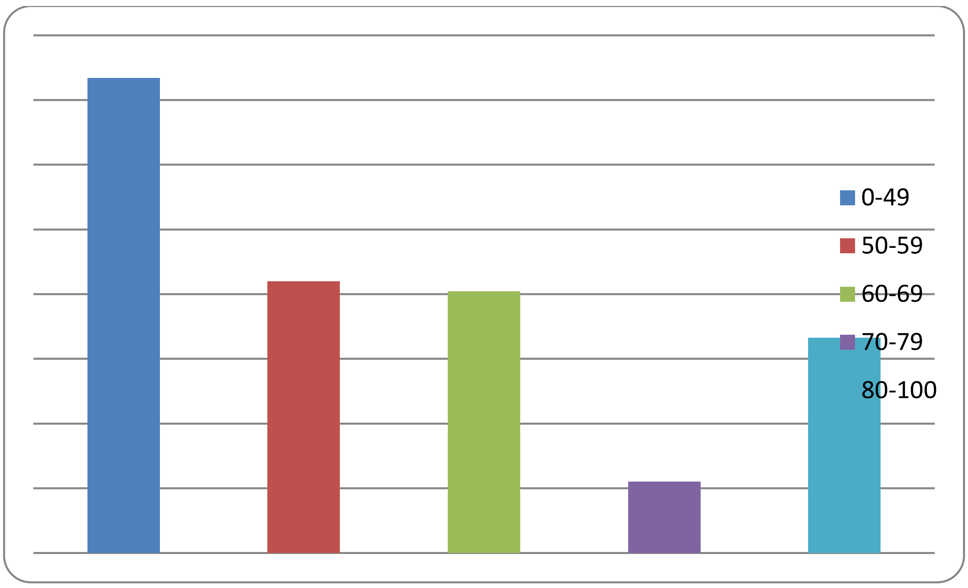

FIGURE 1 : Bar chart on students' marks on test 1, March, 2014 Test 1; Mean = 57%

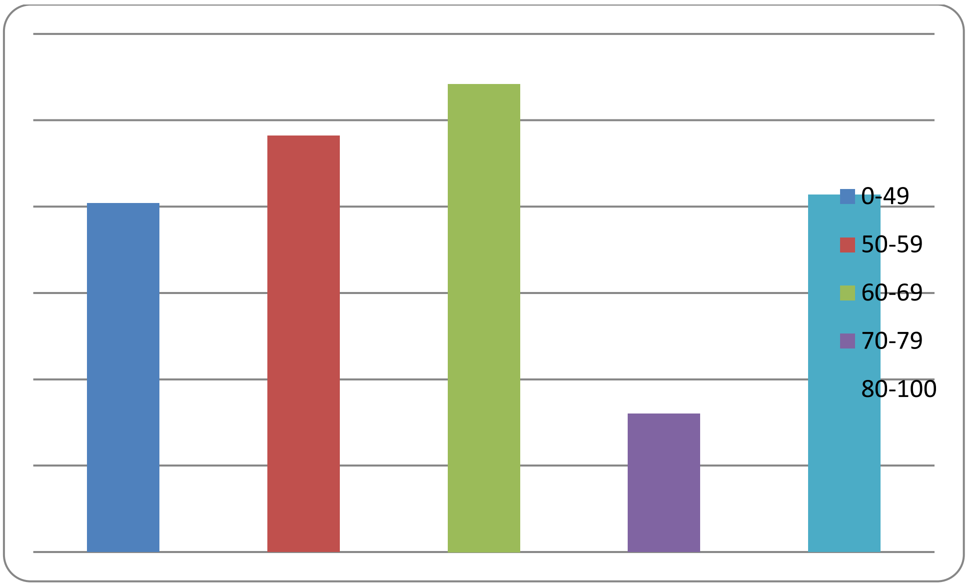

FIGURE 2: Bar chart on students' marks on test 2, October, 2014;

Test 2 Mean = 62%

In the two bar graphs (Figs 1 and 2) one notes a group of students; those with distinctions scoring 80% and above in test 1 and test 2., These students were consistently good and do not seem to have studied the course at all. These students showed no problems with MR. Also the distribution of test 1 is positively skewed whereas in test 2 the distribution is approaching the symmetry of the normal distribution. Nonetheless, the graph shows a bimodal distribution in test 1. The two modes are fail (37%) for the first group and first class (17%) for the other group. The test 2 graph shows significant improvement of student performance from semester 1 to semester 2. The failure rate also has decreased by 17% to 20% and there are more distinctions.

In order to understand more on this phenomena, qualitative analysis was done with the help of interviews and questionnaires.

Qualitative analysis: Interview data

This analysis is based on interviews with some three Maths Routes students; Mapula, Daphne and Kevin (not their actual names).

Interview with Mapula (28 years of age)

Mapula last did mathematics in 2000 and it was not at matric level. I showed her the following visual (Fig. 3) and asked her to tell me the length of the line.

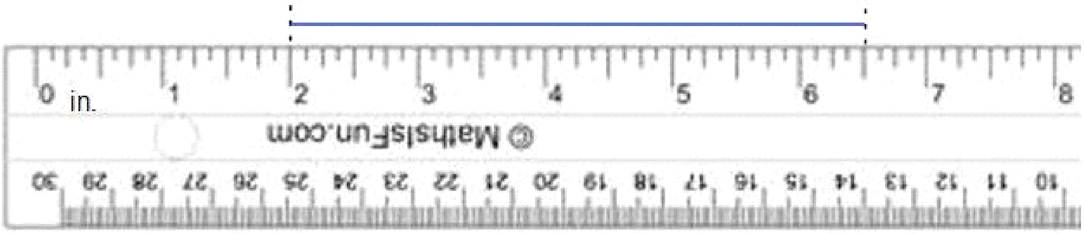

FIGURE 3: Finding the length of a line

Researcher: What is the length of the line?

Mapula: The length of the line is $6^{1}/_{2}$

Researcher: But look it only starts at 2, not 0…

Mapula: Oh then I figure it out, its $5^{1}/_{2}$

Researcher: Show me how you got that

Mapula: I start counting at 2,

(She then says aloud 1, 2,3,4,5, $5^{1}/_{2}$)

It is only when I discuss with her that 2 is the starting point of the line so it's the zero point that she is able to measure the line properly (4½).

Interview with Daphne (about 45 years old)

This student had never passed a Maths Routes test. She was last at school in 1987.

Researcher: Yes Daphne, tell me if you are having any difficulties with Maths Routes or not…

Daphne: Maths was not my strong subject, also my attitude, I hated maths, Maths is my only hard subject the others are manageable.

Researcher: Yes…

Daphne: After having worked for so long (as a workshop attendant), my mind was relaxed, this is a different kettle of fish, I was not adequately prepared when I registered for this course. This place can be very lonely and depressing, the huge age gap… you are sometimes taught by people your age or younger, you find no one to baby sit you.

Researcher: Ehh Daphne, go on

Daphne: Everything is difficult in Maths. In school we did arithmetic ,during our time there was much toy-toying at the townships. I attended … Girls' High in ... Boys came to our school to disturb our learning. We did not cover the syllabus.

(I then decided to give her a typical Maths Routes fractions task)

Researcher: Daphne would you order the fractions ; 0.8; 8and 0.08 from smallest to largest?

Daphne: is the same as 8. So is bigger than 0.8

Researcher: How is that?

Daphne: There is an 8 and a 1, so its 8…

This shows the student has misconceptions on the number concept and fractions; very central ideas in Maths Routes.

Interview with Kevin (22 years of age)

Kevin did Maths Lit in 2013 and was awarded Level 5.

Researcher: Tell me about Maths Routes

Kevin: Some of the things we do in Maths Routes are there in Maths Lit but are solved in different ways. Pure Maths methods are used and I get confused.

Researcher: Yes…

Kevin: … and also graphs, such as cumulative frequency graphs in Maths Lit we did not do them. But probability is easy and tree diagrams … it's the same as Maths Lit. Conversion of units is easy we did in Maths Lit as well as fractions.

Researcher: Yes…

Kevin: Volume and area is also done in Maths Lit but here the level is higher, so it is difficult.

Researcher: Yes…

Kevin: I do not understand notation. makes statistics very hard for me. I can't understand the formula

Researcher: Find the area of this figure (Fig. 4)

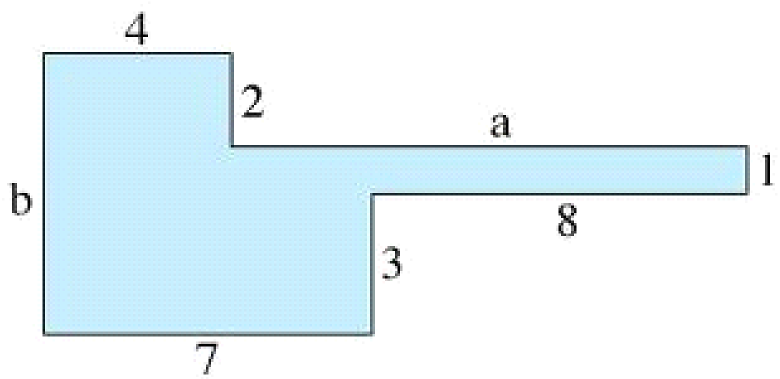

FIGURE 4: Perimeter and area of compound figure

Having split the shape into three rectangles under my motivation, Kevin said a=11. So the area of the thin rectangle is 11x8. When I looked showed him that it must be either 11x 1 or 8x1. He was really perplexed! I looked curiously at him.

Kevin: So if we are saying its 11x1, where did we use 8?

These three students' difficulties illustrate clearly that while instructors may want to go very far with the mathematics content in terms of breadth and depth in Maths Routes it is quite risky to over-assume what students know as some students really have a great deal of difficulties with many basic mathematics concepts which at the same time other students regard as time wasting.

Yet these concepts are some of the most important in mathematics, if teachers cannot acquire these concepts during Initial Teacher Education it really might be seen as reckless on the part of us teacher educators to graduate them without ensuring that they understand them. These key concepts are also critical because they help students to understand other concepts as mathematical concepts are relational in nature.

Qualitative analysis: Questionnaire data

Some of the students' responses on their reflections on MR are tabulated below.

Table 3: Some of the students' open comments on the mathematics routes course

Student	Comments
1.	1. I did Maths Matric and got level 3 in 2013. It was very bad but I did not hate it. In Mathematical Routes I found the part we have to find perimeters and areas difficult because I do not understand these. I believe since I am a senior primary teacher, it would be better if for every lesson or topic formulas are introduced and broken down so that we can know them and make it easy to use in the future.
2.	I did Maths Lit, 2013. Level 6. What I found difficult when Dr Makonye introduced the concept of bases. And also Dr Leshota was lecturing data handling and fractions. What I fund easy was some parts on data handling topic, because I was familiar with some concepts as I dealt with them last year. What I found not necessary was for everyone to do the course, it should be for people who will be teaching maths as it forms a basis for them.
3.	I did Core Maths 2013 and got level 6. The number bases are difficult as well as number theory (i.e. factors, HCF, LCM ,prime numbers, etc.). Fractions were easy but the course must include algebra and trigonometric functions.
4.	I did Maths, 2013. Level 6. Shapes and lines were difficult. Fractions, statistics and probability quite easy. These were also most enriching. Shapes and ancient numeration systems and long lectures are unnecessary. **** I think Maths Routes should be split in two. Maths Routes (pure) and Mathematical Routes(Maths Lit). this is because some topics are relevant for the foundation phase teachers and other topics are relevant for FET teachers.
5.	I did Maths [in] Matric and got level 5 in 2013. Probability, surface area and volume were difficult for me other topics were easy.
6.	Maths, 2013 level 6. I found bases difficult.
7.	I did Maths Lit, 2013. Level 6. To me Mathematical Routes was not easy at all. The things that I found most difficult were stem and leaf plots, I don't understand it even

	now. Fractions were not that hard because I did them at Matric and other staff that include Maths Lit.
8.	I did Maths Lit, 2013. Level 5.I found probability to be a bit challenging. No more material must be added to Mathematical Routes it will become too complicated.
9.	I did Maths Matric in 2012. Data handling was easy. Fractions are important but bases and algorithms is a waste of time. Some topics must be taken out as they are not necessary and not all of us will be maths teachers. Some topics are not necessary at all and are time consuming.
10.	Maths level 5. I found attending lectures and tutorials a waste of time.
11.	Maths level 5, 2013. Bases were confusing. Mathematical Routes for me is basic or a foundation for mathematics and maths lit so everything is actually significant.
12.	Maths, 2013 level 6. Mathematical Routes is boring and easy, I found bases … very interesting. What I found difficult is use of fraction visuals.
13.	Maths, 2012 level 3. Measurement concepts are hard for me, the methods that were used were different from what I in my experience of schooling. The course must include algebra.
14.	Maths, 2013 level 5. What I found difficult was volume. What I found most important was to teach and explain fractions. What needs to be included is explanations of x solvings.
15.	Maths, 2012 level 5. I found bases difficult at the beginning of the first semester. I found most stuff enriching.
16.	I did Maths Lit, 2013. Level 6. The way of teaching fractions to small learners I found most enriching.
17.	I did Maths, 2013. Level 3. What I found most difficult were the basses. Geometry was easy. Statistics were most enriching but fractions were unnecessary. I don't think anything should be included more in this course, it is perfect as it is.
18.	I did Maths Lit, 2013. Level 6. Bases, box and whisker plots and frequency density were difficult. Probability, surface area and perimeter were easy. Must include financial maths.
19.	I never did Maths last did it in Standard 8. At least I understood fractions. I feel people like me who did not do maths must be given more time to work maths problems.
20.	Core maths level 5, 2013. I found bases very hard, they were confusing. I found data handling and fractions enriching. Geometry was useless. The course must have more calculations.
21.	Maths, 2012. The most important courses were statistics, measurements, shapes and probability
22.	Maths Lit 2013; level 5. Actually although I did maths which is close to Mathematical Routes it took me back to my primary maths which I had forgot with the shapes etc. certain concepts were difficult to find the mean, mode, median [particularly when

	value are in terms of x. probability I always had problems with it at school. What I found easy was basically what we did first semester. Number bases I feel like these are not really necessary because we will not teach these concepts. More calculations are necessary such as percentages, calculations as well as financial maths. We do them because we want to pass.
23.	Did Maths, 2013. Level 7. I did not find anything difficult. I enjoyed math. Everything was easy. I enjoyed bases but found it unnecessary as we will not use it again. Also the system (roman numerals. We do not use those systems. Maybe you could include how to teach maths to young learners.
24.	Maths Lit 2013; level 7. Difficult fractions and probability, easy-bases. Nothing was important or enriching.

Analysis of questionnaire data shows that first year mathematics routes students experience the mathematics routes course differently. Students showed and reported different levels of difficulty and success as they engaged with the course. The main topics that appear to be difficult to students were the mensuration topics: perimeter, area and volume (students Mapula, Kevin, A, E and N), number bases (students B, C, F, G, J, Q, R and T). For Mapula it becomes self-evident in that as she could not measure the length of a line segment with a ruler; she does not understand the essence of measurement and so cannot be expected to have any relational understanding of mensuration. Some students felt that number bases must be left out of the curriculum because this topic or aspect is not taught at school. The latter observation is very interesting because working with number bases compels students to reflect on what they do daily when they work in base ten. Number bases thus offer a rich example of unadulterated mathematical thinking which by itself is not very complicated yet critical to reveal the practice of mathematics at a most basic level. Other students found some statistics topics difficult; stem and leaf diagram (G), box and whisker plots (R) and histograms. Some felt the need to include certain topics; financial mathematics (R and V), 'x- solutions' (by which I felt the student was referring to algebraic equation problems) were also requested as well as algebra (N and M). Students P and T said that the methodology of teaching fractions was very rewarding; in particular instead of just 'inverting' students showed joy in learning how manipulation of visuals could help in building relational understanding why 'invert' fractions when dividing. The above analysis shows that the researcher was responsive and inclusive the students' difficulties (Sosibo et al., 2016).

Other students were bored (J). J said 'MR is boring because of its easiness'. She however stated that number bases were enriching. One student offered a solution to deal with the problem of students' variable mathematics achievement. He suggested that there should be two groups; mathematics routes (Foundation

Phase) and MR (FET), which groups would deal with different levels of depth and rigour in the mathematics learnt. This could be a useful recommendation that can help in promoting depth and rigour in this course. This is quite practical as the level of mathematics dealt will be deep enough for the students at the appropriate level. This again would be responsive to learner feelings (Sosibo et al., 2016). However there are ethical issues in teaching students of different abilities separately, particularly in a democratic society in which all students are equal.

DISCUSSION AND CONCLUSION

Findings show that at first, some ITE students struggled with the mathematical demand of the MR course. It is noteworthy that students were taught and assessed on demanding mathematical concepts (Stein et al., 2000; Bloom, 1956; Biggs & Collis, 2014) thereby creating and raising their zones of proximal development (Vygotsky, 1986) on MCK. This teaching and assessment in the MR course was in line with Bernstein's (2000) notions of strong classification and strong framing as well as sustaining high mathematical demand (Stein et al., 2000). Deep mathematical knowledge is what is needed to knowledgeably support learners when they teach mathematics anytime in the future. This research afforded a window to viewing students' thinking processes as they make sense of mathematical tasks. It highlights the challenges that student teachers at times meet with mathematics as they try to learn the subject. If these challenges are not dealt with at ITE then they can easily be passed on to future generations of learners and so perpetuate the vicious cycle of underachievement in mathematics.

As Test 2 data shows, despite the fact that students began the MR course with a set of variable mathematical competencies as shown by the bimodal distribution of Test 1 written in March earlier in the year, one notes that as the course progressed, students began to gain mathematical competence and mathematical content knowledge. Test 2 which was written in July of the same year also shows the gradual amelioration of the bimodal distribution, which had represented different groups of students' performance in maths. The statistics of the end of year results for 2014 (these are not presented in this paper) showed that only about 7 % of the students eventually failed the maths routes course. I also report that at the end of the year Kevin and Mapula passed the course but Daphne did not and had to repeat the course. One reason why there was increased learning and mastery could possibly be ascribed to engaging with rigorous mathematical tasks in tutorial sessions. These tutorials helped to ensure that the depth and rigour of mathematics, and that mathematical demand (Stein, Smith, Henningsen and Silver, 2000) was not lowered, but maintained. The tutorials also focused on students' errors and misconceptions identified in the research; such as those on measuring dimensions of compound 2-D shapes.

So what may be learned in terms of depth and rigour in ITE from this research? The researcher strongly argue that the expectations that instructors set out for their student teachers in mathematics courses; the mathematical demand (Stein et. al., 2000); the learning materials engaged with in pursuit of that demand; how teacher educators engage with the struggling students and are critical in achieving depth and rigour in ITE. Added to this is the role of assessment and setting high expectations for students (Lincoln, 2010). The author has observed that once students realize that they are required to perform at a set level in order to get credit for a course they often are compelled to operate at the expected academic level. One often hears students asking, 'is this task for marks, sir?' If students realize that the tasks are not for marks, they often do not engage in the tasks seriously. So some students require extrinsic motivation (DeLong and Winter, 2002) to draw them to learn. If lecturers' academic expectations for student teachers are low, or a laisse faire attitude (McCauley and Van Velsor, 2003) is allowed, then students will not develop to their full potential in a teaching subject and their ZPD will not be raised. In a laissez faire approach to teaching, anything goes and learners are allowed to do what they like. It is important to establish students' errors and misconceptions (Makonye, 2012; 2015) so that these are used as resources for student teaching; particularly in tutorials where there is more open interaction between the students themselves and tutors. What is required is for the students to experience the ahaa! of learning; when they come to realize how incomplete their knowledge was. So interviews with students on their learning difficulties have potential for setting deeper teaching and learning (Borasi, 1994). If teacher educators do not have interviews they may not appreciate their students' difficulties with mathematics. They cannot hear their students' voices (Davis, 1997). Those voices are vital in fertilizing instruction to help students construct correct mathematical concepts. The students' voices are critical for the growth of a teacher educator's PCK (Davis, 1997) because according to constructivism, students' new knowledge germinates from the old.

RECOMMENDATIONS

To have depth and rigour in ITE particularly in the mathematics subject; the study recommends;

1. That a mandatory foundation mathematics course such as maths routes outlined in this paper be taught to all teachers in training irrespective of what subject they will teach at school. The mathematics content for this course may cover only up to lower secondary; grade 8-9.

2. That the mathematics foundation course must have strong classification and strong framing

3. That to move towards this end ITE institutions must set up a committee to draw a uniform curriculum for this course

4. That ITE institutions benchmark the ITE curriculum and practices of those countries that comparatively do better than others in international comparison tests in various subjects.

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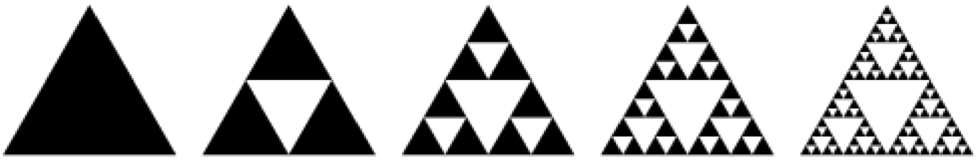
APPENDIX - EXEMPLAR MATHS ROUTES TEST TASKS

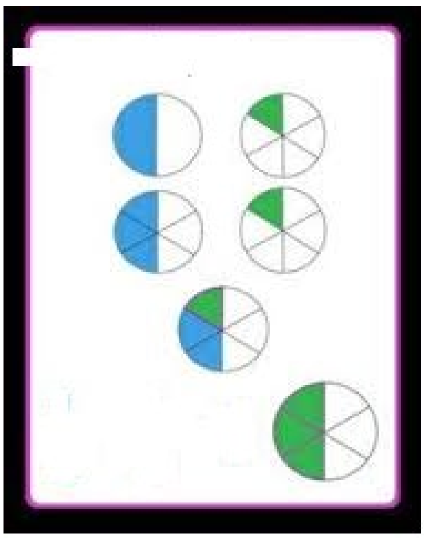
1. Consider the Sierpiński gasket made by progressively cutting pieces of triangle/s from the original black triangle. In the first diagram, we have a full triangle, in the second, one triangle is cut and three black triangles remain, in the third diagram a white triangle is cut from each of the three triangles that remained in the second diagram. The cutting pattern continues in that manner. Find the fraction occupied by black triangles in the 15th diagram.

(Note: The sixth diagram and the others that follow are not shown here.)



2. Brian and Xolile were both saving their pocket money. For every R10 that Brian saved, Xolile saved R20. In other words, Brian saved ½ of the money that Xolile saved. They decided to buy a birthday present for their mother. For every R3 Brian spent on the present, Xolile spent R7; that means Brian paid for $^{3}/_{10}$ of the cost of the present. After they have bought the present, Brian had already spent a third of his savings and Xolile only had R95 left over. What was the price of the present?

3. What does the following visual help to **best** illustrate?



I. $^3/_6$ times $^1/_6$

II. $^1/_6$ divided by $^1/_2$

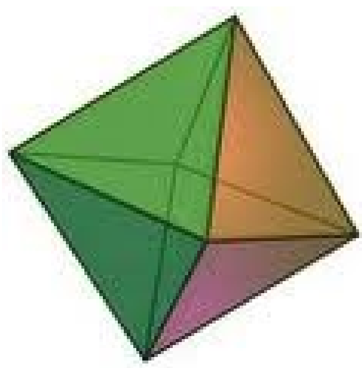
III. $^1/_2$ divided by $^1/_6$

IV. $^1/_6$ plus $^1/_2$

A	II only
B	I and III
C	III only
D	IV only
E	None of these

4. Evaluate $111011_{two} + 10111_{two}$ and give your answer in base two.

5.



An octahedron is a platonic figure with 8 faces. Given that it has 6 vertices, how many edges does it have?

MATHEMATICS ATTITUDES OF LEARNERS IN SCHOOLS LOCATED IN LOW SOCIO-ECONOMIC AREAS

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The study reported on here was carried out as part of an ongoing larger research project. The project seeks to find innovative context-based ways to improve the teaching and learning of Mathematics in high schools located in low socio-economic areas of the Cape Town Metro municipality. To this end, a lesson study is employed as a school-based continuous professional development (CPD) approach. In an attempt to acquire an authentic understanding of the context of teaching Mathematics in schools located in low socio-economic areas, this study explored high school Mathematics learners' attitudes based on five attributes or instrument subscales. These are, personal confidence, subject usefulness, male domain, teacher perception, and the home background. This study also examined the inter-correlations within the identified subscales in terms of learners' attitudes towards the teaching and learning of Mathematics in schools located in low socio-economic areas. The findings showed, for instance, that there was a statistically significant positive association between student's confidence and their achievement scores in Mathematics. To deepen the understanding of the teaching contexts of schools located in low socio-economic areas, future research using a narrative approach may be necessary to capture "stories by learners" and "stories about learners" and, understand what learners do and do not value during the teaching of Mathematics.

Keywords: Mathematics teaching & learning; Students' attitudes; Attribution theory; high-stakes examination; socio-economic areas; Fennema-Sherman Mathematics Attitudes Scale

INTRODUCTION

There has been significant concern about the state of South African students' academic achievements in the National Senior Certificate (NSC) high-stakes examinations. This particularly, in the performance of students in schools located in low socio-economic areas, that is, schools located in previously disadvantaged areas which includes townships and informal settlements (Arends, Winnaar, & Mosimege, 2017; MDG-CR, 2015). For instance, according to the National Strategy for Mathematics, Science and Technology Education in General and Further Education and Training (DBE, 2001), the number of learners who participate and successfully pass Mathematics in Grade 12 is comparatively very low. Moreover, of these low numbers, the proportion of African candidates, whom are predominantly from low socio-economic areas, who participate and succeed is disturbingly low. In addition, learners from schools located in low socio-economic areas tend to bear the brunt of this underperformance since they are more vulnerable to the impacts of repetition and dropping out (Volmink, 2010; DBE, 2001).

One of the concerns faced by the South African government in post-apartheid era is the failure to provide quality Mathematics education for its diverse society of about 55 million people. In accordance with Heyneveld and Craig (1996, p. 13), the word 'quality' in relation to education, in the context of the current study, refers to "both changes in the environment in which education takes place and the detachable gains in learners' knowledge, skills and values". Quality Mathematics education goes beyond knowledge transmission during teaching and learning to include affective and social knowledge in addition to curricula and teaching (Winheller *et al*., 2013). South African children, in general, compared to their more economically-challenged neighbouring countries, are exposed to poorer quality of teaching, particularly in Mathematics at the secondary school level (Howie, 2003; Taylor, 2008). The National Planning Commission (2012) acknowledge that, in the main, most black children receive poor quality education in South Africa. This denies these children access to higher education, and access to decent employment leading to reduced earnings and limited career mobility to those who do get jobs. The current study therefore also seeks to enhance our understanding of the pertinent context that permeates such schooling environments as it relates to Mathematics teaching and learning, particularly, from the perspectives of the learners. This, especially since the predictive quality of NSC marks is not reasonably significant for learners in this context compared to those from schools located in high socio-economic areas (Volmink, 2010). An improved understanding of the students' feelings and perceptions about the factors affecting their own learning of Mathematics will provide critical vantage-positions from which educators and other education stakeholders can introduce appropriate interventions.

International comparisons of educational achievement in Mathematics, for example Trends in International Mathematics and Science Study (TIMSS) conducted in 1995, 1999, and 2002 at grade 8 level, have also shown that South African students performed significantly lower than other participating countries. Although some changes were evident from 1995 to 2002, they were statistically insignificant (Fleisch, 2008; Spaull, 2013). TIMSS shows that between 2002 and 2011, there was a substantial improvement in Mathematics scores for grades 8 and 9, however, this came from a very low base. In addition, South African students' performance still lagged other middle-income countries. Evidence from other reports and studies have also shown that students have also achieved equally poor results in the matriculation examinations (Howie, 2003).

Whilst there are several factors that have been reported pertaining to the performance of students in Mathematics, we argue here that one of the main factors is the students' background, and their attitudes towards Mathematics learning and teaching. Recent lines of research about teaching and learning of school Mathematics have shown that students' attitudes towards Mathematics are very important (Pepin, 2011). These attitudes affect students' achievement in Mathematics. For example, data from Programme for International Student Assessment (PISA) and TIMSS, and Progress in International Reading Literacy Study (PIRLS) showed that self-efficacy was correlated with achievement in Mathematics (Marsh et al., 2006). Learners' Self-efficacy refers to personal judgments of learners' capabilities to organize and execute courses of action to attain designated goals (Zimmerman, 2000; Bandura, 1997). Apart from self-efficacy, perceived usefulness of the subject is also seen as an important contributor to the students' achievement in Mathematics (Marsh et al., 2006).

The affective domain relates to an "individual's experience of feelings or emotions as a result of some external stimuli" (Osler, 2013, p. 36), and is measured by values, attitudes, and perceptions to name a few. Recent research in Mathematics education concur that the features of the affective domain, as much as the cognitive domain, are crucial in understanding high school students' under-achievement in Mathematics (Ashcraft & Ruding, 2012; Boaler, 2015). Research carried out in other contexts (not specifically pertaining to the low socio-economic areas in the Cape Town Metro in South Africa) has shown that there is a significant positive correlation between the students' attitudes in the affective domain and their achievement in Mathematics (Arikan et al., 2016; Ganley & Lubienski, 2016). The aim of this study is to explore high school Mathematics students' attitudes. In addition, the study seeks to examine whether there are any correlations within and across the identified categories in terms of the students' attitudes towards the teaching and learning of Mathematics in schools located in low socio-economic areas. In this study, data was collected using an adapted Fennema-Sherman Mathematics Attitudes Scales (FSMAS) (Fennema & Sherman, 1976). In order to explore high school students' attitudes towards Mathematics, five affective domain categories were used in this study, namely: personal confidence about the subject of Mathematics, usefulness of the Mathematics' content, subject is perceived as a male domain, perception of the mathematics teacher's, and perceptions of home background influence in Mathematics. Therefore, the following research questions are investigated in this study:

- What are high school students' attitudes towards the learning and teaching of Mathematics in schools located in low socio-economic areas?
- Are there any inter-correlations between high school students' attributes, such as for instance, between student confidence and their perceptions about Mathematics teachers?
- Are there any correlations between high school students' attitudes and their achievement scores in a Mathematics?

THEORETICAL FRAMEWORK

In this study attribution theory is used to understand how students interpret their achievements in Mathematics. This theory is used to explore how the learning and teaching environments of students in school located in low socio-economic areas influence their Mathematics achievements. When a student gets a low mark in Mathematics, she/he will attribute his/her lack of achievement to a specific cause, such as personal confidence about the subject of Mathematics, usefulness of the Mathematics' content, subject is perceived as a male domain, and perception of the teacher's poor instruction methods among others (Khedhiri, 2016). The attribution given by the student will affect the way the student engages with other Mathematics-related learning activities. Popham (2005) and Goodykonntz (2011) assert that students' performance in Mathematics is affected by their poor attitudes towards the subject. Weiner's attribution theory (2005) classified students' attributions as internal – this takes place when a student performs well and attributes the good performance to their own ability and hard work. External attribution on the other hand happens when a student performs poorly in Mathematics and attributes this poor performance to external factors, such as, their poor backgrounds (including family) or lack of competence and motivation from teachers or Mathematics being a difficult subject, or their negative attitudes towards Mathematics, or attitudes towards the perceived usefulness of Mathematics in their future careers (Yee, 2010).

METHODOLOGY

Participants and Procedure

This study is part a bigger project which aims is to improve the teaching and learning of Mathematics in schools located in low socio-economic areas. This is done through implementing school-based continuous professional development (CPD) for teachers using a lesson study approach. This study is focused on the Further Education and Training (FET – Grade 10 to 12) phase of secondary schooling for schools located in low socio-economic areas of the Cape Town Metro. All of the three schools participating in the bigger project also participated in this study. A targeted sample of all Mathematics students in the selected three schools participated in the study and no students studying Mathematical Literacy participated in the study.

In this study a higher proportion (two-thirds) of the participants (N = 252) were girls and, only 83 (33%) boys participated in the study. Under the guidance of the researchers, the Mathematics teachers in each of the three grades administered a survey by grade asking students to fill out the survey instrument. Ethical clearance was sought from the Western Cape Education Department while schools, teachers, and the parents of the students granted permission for the study to be carried out. The student respondents executed the survey by indicating the level of agreement or disagreement with each statement item. In addition, participants also provided demographic data such as gender and grade level. The respondents were also asked to provide consent for their educators to provide their final Mathematics marks of the previous grade for the purposes of the research. For instance, the grade 10 learners' marks that are analyzed in this study are those from their grade nine final examinations, in the previous academic year, 2018. The respondents were informed that results emanating from the survey will be utilized for research related purposes only and, that their responses will be treated with optimal confidentiality.

Instrument

In this study data was collected using the adapted FSMAS scale. The instrument was adapted to align with the study's context. The instrument seeks to measure students' attitudes and perceptions towards Mathematics. The instrument consists of five subscales - a confidence scale (C) which measures student's beliefs about their own learning abilities in Mathematics; a usefulness scale (U), that measures student's perceptions about the benefits of learning Mathematics; a scale that measures student's beliefs about Mathematics being a male domain (M), a teacher perception scale (T) which measures student's attitudes about their teachers, and a scale measuring student's perception of the influence of their home background on learning Mathematics (H). Each of the subscales C, U, M, and T contained 6 items (4 items for the H sub-scale) and items included statements like "Math is a worthwhile, necessary subject", measured on a 5-point Likert scale (A = Strongly agree, B = Agree, C = Not sure, D = Disagree, and E = Strongly disagree).

DATA ANALYSIS

The statistical analyses of the data were carried out using both Microsoft Excel 2016 and IBS SPSS 2017 Version 25 statistical software. The data in this study are measured using a Likert scale. Each subscale attribute is composed of a series of six (four for the Home Background subscale) Likert-type items that were combined into a single composite score during the data analysis process. Each response (A to E) for every item was converted into a numerical score from 1 to 5 depending on whether the item is identified as positive or negative. That is, for a positive item such as "Maths is a worthwhile, necessary subject", the responses are scored using the following one-to-one correspondence *viz.* "A = 5, …, E = 1", and then the order is reversed for a negative type of item. Thus, the maximum score is 20 for the Home background subscale and 30 for the other subscales. The data in this study was therefore analyzed at the interval statistical measurement scale, as is suggested by (Boone & Boone, 2012).

Validity & Reliability

Whenever items are used to form a scale they need to have internal consistency, that is, the items should all measure the same thing. This means that they should be correlated with one another or have internal validity (Bland & Altman, 1997). The Fennema-Sherman Mathematics Attitudes Scales used in this study has a five Likert-scale as outlined above. Internal consistency reliability estimates, and associated statistics of the overall scores and the sub-scales were analyzed and are reported below in Tables 1 to 4. Validity and reliability are two fundamental elements in the evaluation of a measuring instrument (Tavakol & Dennick, 2011). Whereas, validity is concerned with the extent to which an instrument measures what it is intended to measure, reliability is concerned with the ability of an instrument to measure consistently (Tavakol & Dennick, 2011). Cronbach's alpha is the most widely used objective measure of reliability (Hlalele, 2018; Forootan, Tabatabaeefar, Maghsoodi, Ardeshiri, Fatemi, & Rahimzadeh, 2014; Huang, 2011; Tavakol & Dennick, 2011; DeHaven & Wiest, 2003; Bland & Altman, 1997). Table 1 shows the mean and variance of the overall scores of the five sub-scales in the questionnaires.

Table1: Adapted FSMAS Scale Statistics.

Mean	Variance	Standard Deviation	Number of Items
111.00	143.829	11.993	5

The results in Table 2 depict a reliability index of 0.679, and in Table 4, the alpha values for the subscales range from 0.524 for the Home Background scale to 0.772 for the Confidence scale.

Table 2: Adapted FSMAS Internal Reliability Statistics

Cronbach's Alpha	Cronbach's Alpha Based on Standardized Items	Number of Items
.679	.695	5

The Cronbach's alpha values reported in Table 3 estimate what the Cronbach's alpha would be if we removed a sub-scale. If, for instance, the Male Domain scale is removed, then the Cronbach's alpha of the overall scale would increase from 0.679 to 0.732. Thus, we could remove the item from the scale and trade it away as a proxy for higher reliability however, any value in the range 0.6 to 0.7 is generally considered acceptable. Moreover, an alpha of 0.6 is referred to as a common threshold for sufficient values of Cronbach's alpha (Hlalele, 2018).

Table 3: Adapted FSMAS Subscale-Total Statistics.

Subscale	Scale Mean[a]	Scale Variance[a]	Corrected Subscale-Total Correlation	Squared Multiple Correlation	Cronbach's Alpha[b]
Confidence	88.43	85.386	.530	.379	.581
Home Background	95.48	104.035	.564	.343	.591
Male Domain	86.31	112.310	.202	.090	.732
Teacher Perception	87.50	93.725	.569	.353	.570
Usefulness	86.30	101.692	.387	.186	.649

a. Scale statistics if the subscale is deleted
b. Cronbach's alpha if the subscale is deleted

Confidence intervals at the 5% significance level are also reported in Table 4. The results from the table show, for instance, that students from schools located in poor socio-economic areas are on average confident about their mathematical abilities, *22.57 (95% CI:22.05 – 23.09)*. As is also depicted in the table, the mean scores are statistically significant ($p < 0.05$) for all the subscale attributes.

Table 4: Subscale Statistics, Confidence Intervals and Subscale Reliability, $N = 252$.

Subscale	Min	Max	Mean (95% CI[a])	Standard Deviation	Cronbach's Alpha
Confidence	6	30	22.57 (22.05 - 23.09)**	4.181	.772
Home Background	4	20	15.52 (15.18 - 15.87)**	2.785	.524
Male Domain	6	30	24.69 (24.21 - 25.17)**	3.870	.562
Teacher Perception	6	30	23.51 (23.08 - 23.94)**	3.461	.636
Subject Usefulness	6	30	24.71 (24.25 - 25.16)**	3.670	.662

a. 95% Confidence Intervals for means
** Statistically significant, $p < 0.05$

Table 5 is a correlation matrix, displaying how each subscale correlates to all the other subscales. As can be seen in the table, all inter-subscale correlations were in the expected positive direction. The estimated correlation between the subscales was lower than the internal consistency of each of them measured by the Cronbach alpha. This indicates that each subscale of the adapted FSMAS scale had an ability to measure only a single attribute, as is desired (Forootan *et al.*, 2014).

Table 5: Inter-Subscale Correlation Matrix

Subscale	Confidence	Home Background	Male Domain	Teacher Perception	Usefulness
Confidence	1.000	.482	.081	.423	.332
Home Background	.482	1.000	.280	.499	.299
Male Domain	.081	.280	1.000	.133	.085
Teacher Perception	.423	.499	.133	1.000	.307
Usefulness	.332	.299	.085	.307	1.000

RESULTS & DISCUSSION

Linear Association Between Subscales

The Pearson product-moment correlation coefficient, denoted by r, is a statistical measure of the strength of a linear association between two variables of interest. Often, it is important to know whether a relationship exists between two variables, e.g. between personal confidence in a subject and achievement scores for learners (Steyn, 2002). In this study Pearson product moment correlation coefficients measuring the linear relationship between the Confidence subscale and other subscale show a positive correlation across all the other subscales. The correlation coefficients range from $r =$

0.1 to $r = 0.5$. According to the analysis in Table 6, there is virtually no correlation between student mathematical confidence and their perceptions of the subject being a male domain. Whereas, there exists a statistically significant linear relationship ($p < 0.01$) between confidence and the other subscales. For instance, there is a weak-positive relationship between FET students' confidence and their perceptions of the Mathematics teachers ($r = 0.514$, $p < 0.01$), this is also shown in Figure 1 below.

Table 6: Pearson Correlations across Subscales & Statistical Significance, $N = 252$.

		Confidence	Teacher Perception	Home Background	Male Domain	Usefulness
Confidence	Pearson Correlation	1	.514**	.500**	.103	.362**
	Sig. (1-tailed)		.000	.000	.051	.000

** Correlation is significant at the 0.01 level (1-tailed).

Students who exhibit high levels of confidence in their mathematical abilities tend to also have positive perceptions of their Mathematics teachers. It can also be seen from the displayed R-squared (0.264) value that this is a weak linear relationship.

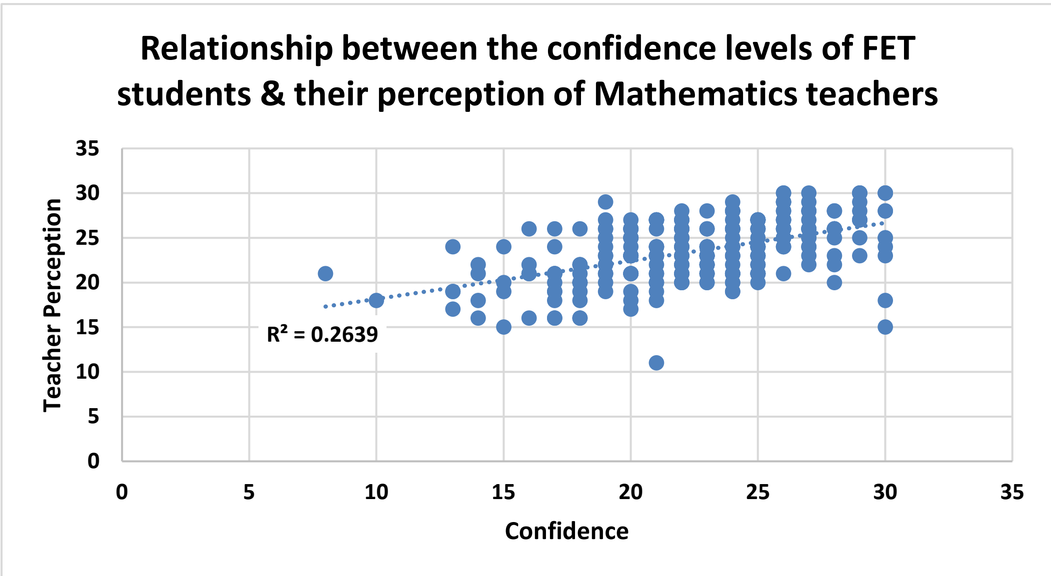

Figure 1: A scatter plot of student confidence levels and teacher perceptions.

The study also tested the supposition that male students tend to have more confidence in their mathematical abilities than their female counterparts, an independent samples t-test was performed, in other words, "males and females have the same confidence levels in Mathematics". The results of the test were not significant ($p = 0.35$) and concluded that there was no difference between males and females in terms confidence levels in Mathematics.

Analysis of student performance

Data for the student's performance marks was only available for a subset ($N = 123$) of the 252 respondents and so all the reported results of student's performance only relate to this subset.

Student Performance Marks vs Subscale Attributes

Table 7 shows that there is virtually no association between student performance in Mathematics and their perceptions of the Mathematics teachers.

Table 7: Correlations with performance marks & Statistical Significance, $N = 123$.

		Teacher Perception	Teacher Perception	Confidence
Mark	Pearson Correlation	1	.091	.479**
	Sig. (1-tailed)		.158	.000

**. Correlation is significant at the 0.01 level (1-tailed).

Whereas, there is a statistically significant weak-positive linear relationship between the student's confidence in Mathematics and their performance marks ($r = 0.479$, $p < 0.01$), see also figure 2.

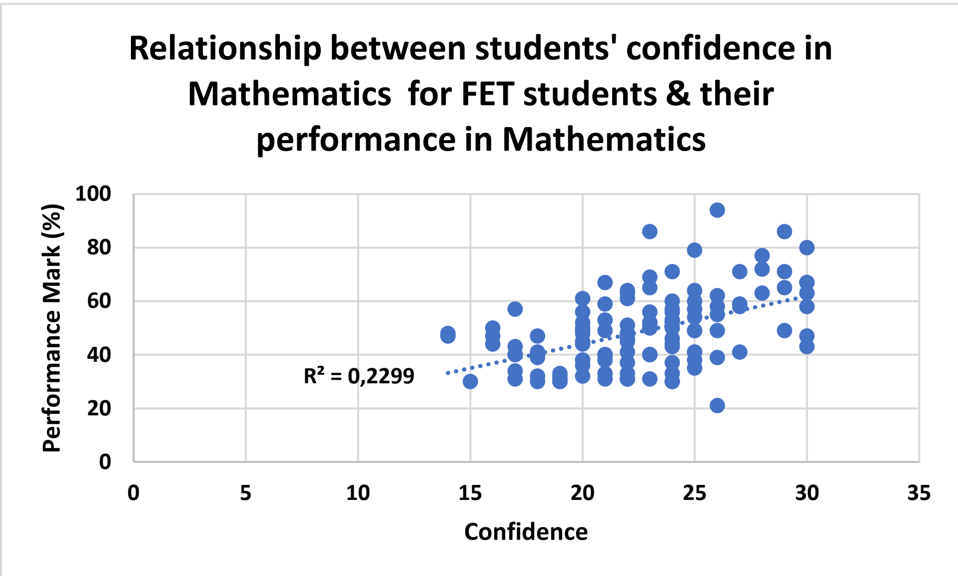

Figure 2: A scatter plot of student confidence levels and performance mark.

CONCLUSION

This study explored high school Mathematics students' attitudes in school located in low socio-economic areas of the Cape Town Metro. Whilst, the study of students' attitudes towards Mathematics is not entirely new (Arikan et al., 2016; Ganley & Lubienski, 2016), this study has also shown that correlations exist between the sub-scales of attitudes studied. In other words, learners' attributes relating to personal confidence, subject usefulness, male domain, teacher perception, and student's home background are not mutually exclusive. One of the important findings of the study is that a significant correlation exists between students' confidence scores in Mathematics and their achievement scores in Mathematics. Since this study is part of the bigger project aimed at improving Mathematics in disadvantaged schools, building students' confidence is thus considered part of improving Mathematics results. The question for further research therefore becomes, "what should be done by teachers, schools, and other stakeholders to build mathematical confidence among students". This paper recommends that school-based CPD of Mathematics teachers should focus on strategies of developing the affective domains of students, in addition to developing the teachers' mathematical knowledge of teaching. Future research needs to use the narrative approach to capture "stories by students" and "stories about students' to understand what students do and do not value during Mathematics teaching, and about Mathematics as a subject at school.

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EXPLORING STUDENTS' USAGE OF AN ONLINE INTERACTIVE PROGRAM

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This study explored students' usage of an online interactive program which was designed to improve their mastery of mathematical procedures. The aim was to examine students' mastery of mathematical procedures through repetitive practice of procedures relating to simplification of exponents. A purposive sample of 42 first -year extended engineering students was selected. This quantitative and qualitative study was conducted in one of South Africa's public universities located in the Gauteng province. Sources of data for the study included students' online reports and an open-ended questionnaire. The findings indicated that some students made use of the program successfully by showing mastery of mathematical procedures. However, other students only attempted exercises requiring skills with which they were already familiar and comfortable.

Keywords: usage, interactive, mastery, procedures, online program

INTRODUCTION

Engagement with online interactive programs happens along existing presupposition that education institutions integrate technology into mathematics learning in order to enhance deep understanding. For example, the standards from National Council of Teachers of Mathematics (NCTM) (2000) were set to promote different approaches to mathematics learning, including the integration of technology. The NCTM's expectation on the use of technology by its members to enhance the teaching and learning of mathematics is in line with Goal 20 of the Department of Basic Education's (DBE) action plan to 2019, which stipulates "*to increase access amongst learners to a wide range of media, including computers, to enrich their education*" (2015, p. 6). In response to this call for integration of technology, the university where this study was conducted encouraged lecturers to embrace the use of technology in teaching mathematics. Subsequently, an online interactive program for mathematics learning was put in place and made accessible to students. The purpose of the program is to improve extended engineering students' mastery of mathematical procedures through practice of procedures for solving a particular class of problems. Extended engineering students are students who complete an undergraduate engineering qualification over 4 years instead of the normal 3 years due to their lower admission score.

Mathematics is considered a challenging discipline to learn through online interactive programs (Trenholm, Alcock & Robinson, 2015), yet efforts are being made to foster mathematical literacy through online programs. According to Foster (2018), there is an increased interest in creating ways to support the mastery of mathematical procedures via interactive programs. National Association of Mathematics Advisors (NAMA) (2016) and National Centre for Excellence in the Teaching of Mathematics (2016) stress that practicing mathematical procedures online can serve as a route to developing both procedural mastery and conceptual understanding. Therefore, there is a possibility that practicing procedures online can influence students' mathematics learning. Including online interactive programs into mathematics learning can enhance deep understanding and thereby address learners' misconceptions and the errors they make when tackling mathematical problems.

In this paper, researchers explored students' usage of an online interactive mathematics program intended to help students master mathematical procedures by practicing procedures for simplifying exponential expressions. Researchers explored students' usage by considering the mastery score achieved by each student and the relative speed applied on practicing a particular mathematical procedure. The topic of exponents was chosen since exponents are taught to first year extended engineering students during Term 1. Term 1 extends from middle January to end March. The relationship between the speed taken on solving a specific exponential problem and the mastery of the necessary mathematical procedures was examined. Mastery implies "knowing when and how to apply a procedure and being able to perform it efficiently, accurately, and flexibly" (National Council of Teachers of Mathematics, 2014, p. 1). Mastery is assessed using the 4 difficulty levels, namely, levels 1, 2, 3 and 4. The findings of this study may make some contribution towards understanding the way students use online programs to master mathematical procedures.

This paper begins by exploring literature underpinning the study. We will then go on to describe the theoretical framework of the study. Then lay out the methodology used in the study. The last sections will present the findings and a discussion of the implication of the findings to future research into this area. Finally, the conclusion gives a brief summary and critique of the findings.

RESEARCH QUESTION

How do first-year extended engineering students use an online interactive program to master mathematical procedures through practicing procedures for simplifying exponential expressions?

Objectives of the study

- To explore students' usage of an online mathematics interactive program.
- To examine students' mastery of mathematical procedures through their repetitive practice of particular procedures.

THEORETICAL FRAMEWORK

So far, this paper has focused on the introduction to the study as well as the research questions. This section will outline a summary of the theoretical framework underpinning the study. It is worth noting that the theoretical frameworks used in mainstream educational research are also appropriate for technology-facilitated learning (McDougall & Jones, 2006). The study was informed by the notion of micro-modelling as outlined by Watson and Manson (2006). They argue that students' response to mathematics exercises has something in common with mathematical modeling (2006). Mathematical modeling has commonly been assumed to mean the process of translating between mathematics and the real world in both directions (Blum & Ferri, 2009). For Watson and Manson (2006, p. 93) micro-modeling means

> "the processes of trying to see, structure, and exploit regularities in experiential data, so that learners are thus exposed to mathematical structure affording them enhanced possibilities for making their own sense of a collection of questions, or an exercise"

It is the structure of the exercise as a whole, not the individual items, that promotes mathematical sense making (Watson & Manson, 2006). Furthermore, the modeling perspective described by Leash and Doerr (2003) draws on students' natural desire to engage with and make sense of experiential data (exercises). Watson and Manson (2006) agree that it is natural for students to test their ideas as much as they need to for personal conviction or make continued exploration possible.

Watson and Manson's (2005) notion of micro-modeling is in line with a cognitive perspective of mathematics learning which views knowledge as an entity that is acquired in one task setting and applied to another setting. In this study, the assumption is that students can acquire knowledge in one exercise practice of a particular procedure and apply the knowledge gained in that procedure to another exercise.

Researchers used these ideas to analyze students' usage of an online program to master mathematical procedures with regard to simplification of exponents. Students should be able to see and structure the mathematical procedure for simplifying an exponential expression in a way leading to correct solution. They must be able to explore possibilities to make sense of the mathematical procedure to simplify the expression.

LITERATURE REVIEW

Mastery of procedures

Morrone, Harkness, Ambrosio and Caulfield (2004), wrote that students who are mastery- oriented value the process of learning and seek to acquire new knowledge. On the other hand, Foster (2018) points out that achieving mastery in mathematical procedures is fundamental to students' mathematical development. The assumption is that achieving mastery in mathematical procedures might give students power to tackle more complicated mathematics problems (Foster, 2016). According to Foster (2018), the usual way to develop mastery of procedures is through repetitive practice of particular procedures. Lemov, Woolway, and Yezzi (2012, p. 36) contend that "automating skills frees up mental capacity for being creative". Sinclair, Renshaw and Taylor (2004) claim that online interactive programs have been shown to enhance procedural skills and improve certain critical thinking skills. In other words, such programs could assist students to focus on the details of a procedure in a way that supports development of critical thinking skills.

Commenting on procedural practice, Bhakat and Kumar (2014) argue that practicing procedures for solving a particular class of problems is not always a complete way of learning mathematics. Such a premise is consistent with the findings of Watson and Mason (2006) who found out that learning does not take place solely through students observing some patterns in their work. Nonetheless, in a high-stake examination culture, where procedural skills are perceived to be the most straightforward measures to assess outcomes of examinations, schools and teachers are likely to feel constrained to prioritize the development of mastery of procedures over the other aspects of learning mathematics (Office for Standards in Education, 2012).

In this study, researchers argue that a mathematics exercise genre of online interactive program might be capable of addressing mastery of mathematical procedures and subsequently develop learner understanding. The program aims to generate plentiful exercise practice as students tackle specific mathematical problems. The program provides extensive practice of a well-defined mathematical procedure which is often embedded within rich mathematical exercises that are aligned to extended engineering curriculum. In the study, rich mathematical exercises refers to exercises that encompass the curriculum criteria of the university's extended mathematics qualification.

The section that follows moves on to discuss the features of the online interactive program used at the university where the study was conducted.

The online interactive program

As mentioned in the introductory section, the objective of the program is to help students develop mastery of mathematical procedures through repetitive practice of procedures for solving particular class of problems. The program automatically generates unlimited exercises for students to gain practice. An exercise is generally understood to mean a single object consisting of individual questions seen as elements in a mathematically and pedagogically structured set (Watson & Manson, 2006). In this paper, the term exercise is used to refer to a mathematical problem that require the use of a particular procedure for solving the exercise. During exercise practice all interactivity is mediated by the program. Students attempt varied mathematical exercises on their own time and place. Automatic marking is performed and immediate feedback is provided after the student typed and submitted the answer to the specific exercise solved. Feedback is considered a distinguishing characteristic critical to influence learning and can be used to close learning gaps (Trenholm, Alcock & Robinson, 2015). Toping (1986) found that high mathematics improvement and achievement are associated with an average feedback timing of between three to five seconds.

The program checks students' answers instantly and displays whether the student's answer is correct or incorrect. Thereafter, a step-by-step explanation of the solution is presented whether or not a student captured a correct or an incorrect answer. According to Jacobs (2005), such explanations enable students to recognize and correct errors as well as to build confidence when they give a valid response. Students have options of trying a similar exercise or proceeding to the next exercises of more difficulty. This program also has a student dashboard so that students can set their own goals and track their progress.

Exercises given to students have to be of the correct difficulty level. In the program, selected mathematical exercises are tailored according to students' individual ability, and they become more difficult as students improve. This means that students are provided with more difficult exercises as they progress well. Tailoring mathematical exercises according to varying difficulty levels improve students' competence and confidence (Kostons, Gog, & Paas, 2010). The program's difficulty levels are denoted as mastery levels. Students' are assigned mastery levels as they progress through different difficulty levels. The levels range from level 1 to 4. Level 1 contains some revision exercises as well as exercises that test simple mathematics for the type of an engineering student under study. Level 1 is the lowest level while Level 4 is the highest level for achieving a mathematical procedure. Level 2 represents mid-way to achieving mastery while level 3 is closer to achieving mastery.

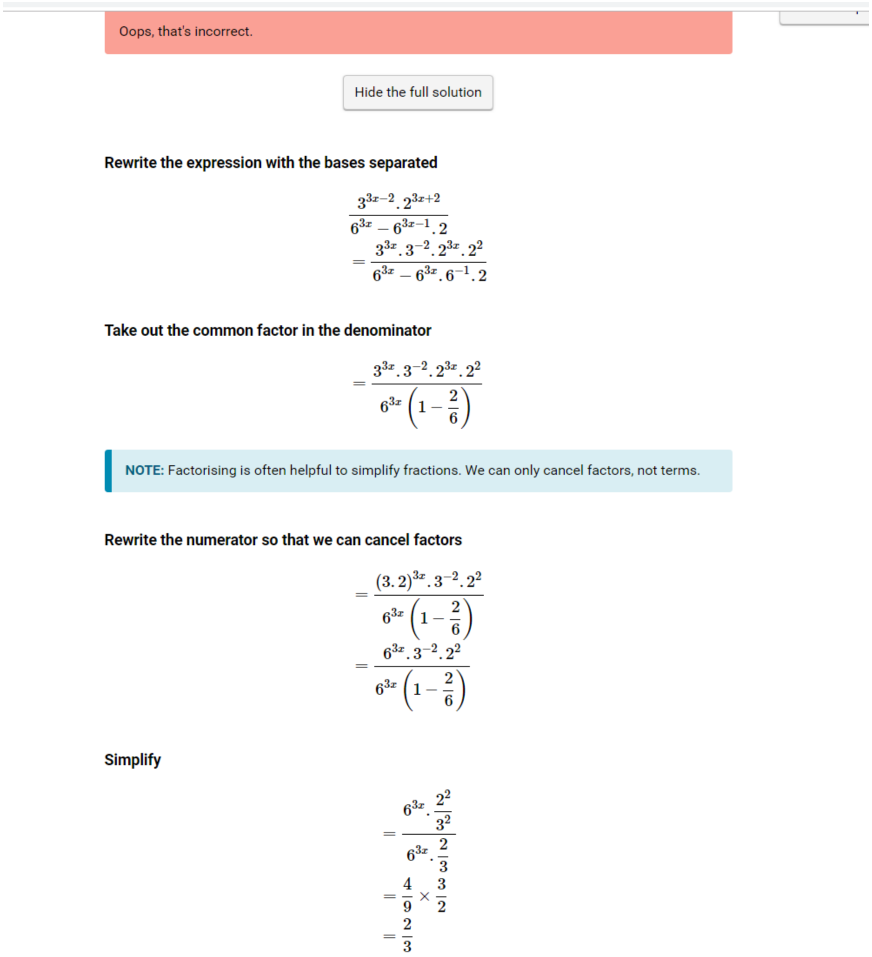
Furthermore, a teacher dashboard is obtainable so that teachers can track the progress of individual students and the whole class. On the teacher dashboard, teachers are able to see, amongst others, the number of exercises each student has attempted, the average speed and the mastery levels. A teacher dashboard enables teachers to see the exercises the student is struggling with, because the program indicates whether the student's answer is correct or incorrect. Also, a dashboard provides teachers with an opportunity to see how quickly students are working by displaying the relative speeds. A negative speed indicates that the student is working slowly while a positive speed indicates that a student is working faster in comparison with his peers. A teacher dashboard also contains filters that can be used to filter specific topics and sections to be completed by students.

In this study, researchers asked students to practice exercises relating to the topic of simplification of exponents. Watson and De Geest (2014) hold the view that for mathematics practice to be effective, the practice must systematically focus on small elements and be purposeful. Students were asked to complete a minimum of 100 exercises from ten sections of exponents within two weeks. According to Watson and Mason (2006), learning takes place over time as a result of repeated experiences that are connected through personal sense making. Grivokostopoulou, Perikos, and Hatzilygeroudis (2017) note that students should be engaged in exercises that are tailored and adapted according to their knowledge level. As a result, the 100 exercises selected met the curriculum requirements of the sampled students.

The program can display the following exercise, which is part of the first year extended engineering mathematics, for the students to simplify: Simplify without using a calculator: $\frac{27^{n+1}.9^{n-2}}{3^{5n+1}}$ (Give your answer as a fraction). The program presents a step-by-step illustration of possible solution. The solution includes explanations to each step.

In the program the solution to the problem $\frac{3^{3x-2}.2^{3x+2}}{6^{3x}-6^{3x-1}.2}$ was presented as shown on Figure 1 below:

FIGURE 1: A possible step-by-step solution that can be displayed by the online program



From the figure above we can see that the solution is presented in a step-by-step way which enables the students to follow how the final answer was arrived at. After illustrating the correct solution, the system will ask the student if he/she would like to try a similar exercise or go on to the next exercise.

So far this paper focused on the introduction, the conceptual framework and literature review. The following section will discuss a summary of research design and methodology followed in the study.

RESEARCH DESIGN AND METHODOLOGY

This section represents a summary of the descriptions and discussions of the research designs and methodology of the study. For McMillan and Schumacher (2001), research designs are procedures for conducting the study, including from whom, when and under what conditions the data was generated. In this study, the researchers adopted a mixed methods research design to describe and analyze first-year engineering students' usage of an online interactive mathematics program.

Data sources

Data provide an evidential base from which to make interpretations and statements intended to advance understanding and knowledge concerning a research question. Lankshear and Knobel (2005, p. 266) claim that data means "bits and pieces of information found in the environment" that are collected or generated in systematic ways. However, David and Sutton (2004) hold the view that data should not be viewed as that which is out there to collect, but rather what the researcher 'manufactures' and records. This means that researchers can determine what counts as data depending on the questions and the objectives that drive the study and in terms of the theory framing the study and analytic protocols adopted.

For this study, researchers wanted to explore first year extended engineering students' usage of an online interactive program designed to enhance mastery of mathematical procedures. As a result, the research data was drawn from two main sources, namely, students' report scores and students' responses to the items on an open-ended questionnaire. Students' online reports contain information relating to number of exercises completed, mastery levels and average speed. Students' online reports were generated from the program. An open-ended questionnaire containing questions relating to students' challenges and benefits regarding their use of the online program was constructed and administered by the researchers.

The study made use of a group of 42 students out of a population of 109 first-year extended engineering students registered at a certain university in the Gauteng province, South Africa. The study adopted purposive and convenient sampling that assisted researchers to answer the research question: how do first-year extended engineering students use an online interactive program to master mathematical procedures through practice of a particular class of procedures? In this study, the results were generalized within the confines of the study and not within the wider population. The study adopted purposive sampling techniques. In purposive sampling researchers handpick units on the basis of their judgement of the cases' typicality or possession of the particular characteristics being sought (Cohen, Manion, & Morrison, 2010). Thus, the selected students possessed certain characteristics the researchers are looking for. These characteristics relate to the selected group of students using an online interactive program designed to enhance mastery of mathematical procedures as well as being taught by one of the researchers.

DATA ANALYSIS

In this study the unit of analysis was students' usage of an online interactive program intended to enhance mastery of mathematical procedures through repetitive practice of particular procedures. The individual student was the point of focus. To understand students' usage of this online program, researchers used qualitative and quantitative data analysis techniques. Qualitative data were generated from open-ended questionnaires while quantitative data was obtained from students' online reports.

Descriptive statistics (means, standard deviations, bar charts and scatter plots) were used to analyze quantitative data. Descriptive statistics was applied for variables created by the online program. These variables include number of exercises completed, mastery levels and relative speed. Mastery levels are the progressive difficulty levels at which students are working. A relative speed indicates whether a student is working faster or slowly in comparison with his peers. Researchers read through the mastery levels and the relative speeds provided by the program. Qualitative data were analyzed using narrative data analysis. Researchers coded data by reading carefully through data from the open-ended questionnaire. Figure 2 below shows an overview of the number of exercises attempted by the students as well as the mastery level (expressed in percentages) assigned to each student. Mastery levels were converted to percentages by researchers by dividing each student's mastery level by 4 and multiply by 100. The highest mastery level given by the program is 4.

FIGURE 2: Number of exercises attempted by each student and the corresponding mastery

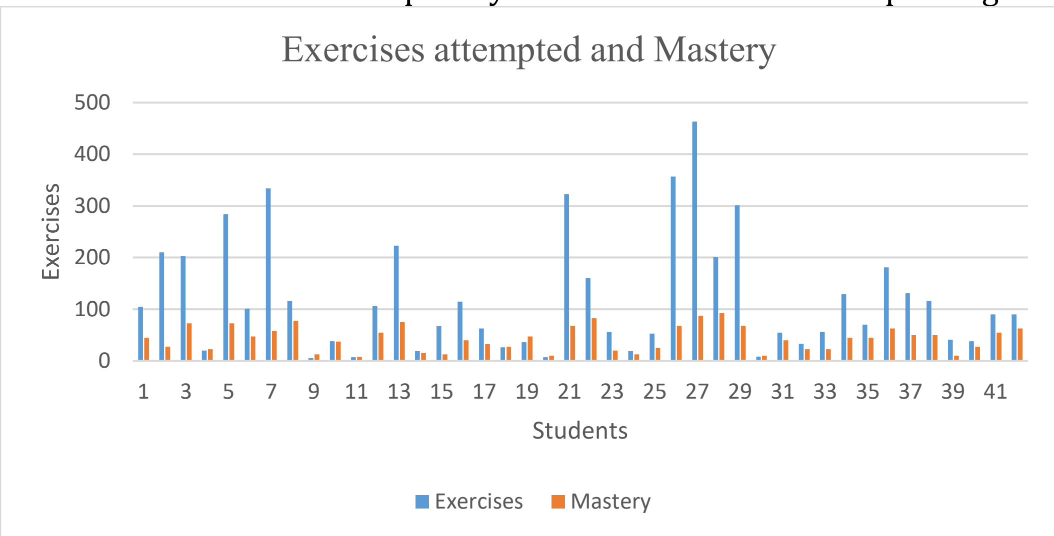
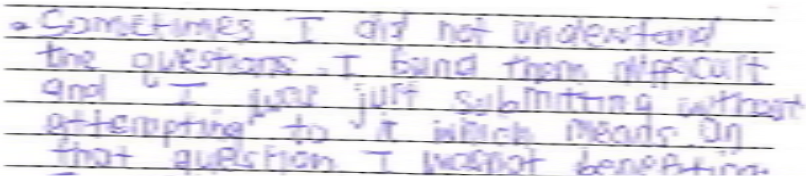


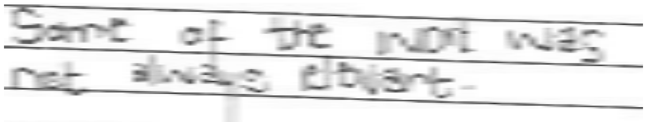
Figure 2 above reveals that some students were only interested in typing answers and did not worry about obtaining correct answers. The assumption is made because the exercises these students attempted were far more than what was expected from them within a period of two weeks taking into account the demands for simplifying each exponential task. Some of the students attempted more than 300 exercises instead of 100 exercises in two weeks while one student attempted more than 400 exercises in two weeks as illustrated in Figure 2. In the program, obtaining a correct answer is linked to mastery of a particular mathematical procedure. Mastery is calculated by the program taking into account correct answers obtained. For example, Student 2 completed 210 exercises and the mastery level percentage is 28%. The acceptable mastery level is a minimum of 50%. The number of students who obtained a mastery level of less than 50% is 22 out of 42 students. The number of 22 students is disturbing since such students did not master the desired mathematical procedure expected by the online program. The 22 students did not progress to the next levels of difficulty. This is supported by the data from the open-ended questionnaire in which one of the students made the following comment:

> "Sometimes I did not understand the questions. I found them difficult and I was just submitting without attempting to it which means on that question I was not benefitting".


- Sometimes I did not understand the questions. I found them difficult and I was just submitting without attempting to it which means on that question I wasnot benefiting

From this student's response, there is a possibility that some students found it difficult to learn procedures for simplifying exponential expressions. As a result, they could not simplify the expressions correctly which lead them typing an incorrect answer. If a student type incorrect answers, the program prohibit such a student to move to exercises of next difficulty level. The program will only produce exercises of the same difficulty level until a correct answer is typed. About 4 out of 42 students wrote a similar comment of submitting without understanding the procedure.

Another student wrote "*some of the work was not always relevant*".


Some of the work was not always elvant-

This student's response might suggest that the student found the exercises unsuitable to the curriculum. An alternative explanation is that the exercises did not meet the student's expectation and might be a result of the interactive program's design. It is also more likely that some students just typed answers because the help provided by the program is insufficient to assist the students to master procedures. As a result, certain students continued typing answers without learning mathematical procedures that are given by the online program. So the program did not provide opportunities for 8 students out of 42 who completed the questionnaire to move to the exercises of next difficulty.

Not obtaining correct answers lead to lower mastery levels since the mastery level is calculated based on number of correct answers exercises completed. Therefore, the practice did not yield the desired results for such students. The purpose of the online program is to enable students to master mathematical procedures through repetitive practice of a particular class of procedures.

Figure 3 below compares students' mastery levels to relative speed:

Figure 3: Students' relative speed and the corresponding mastery levels

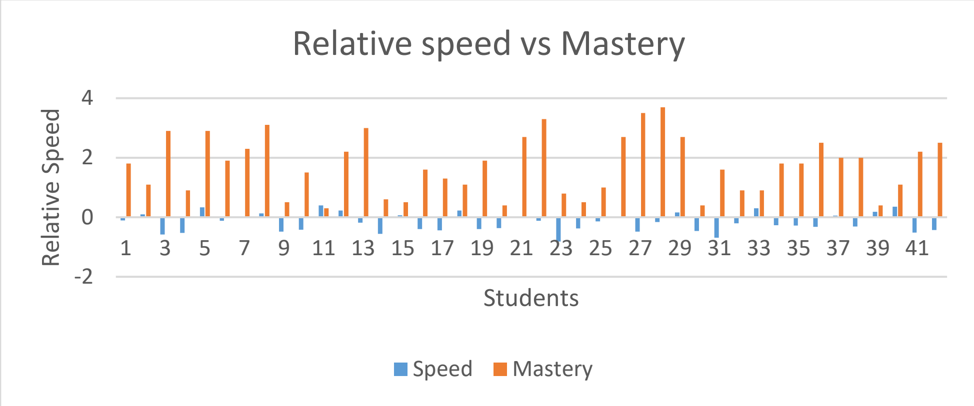
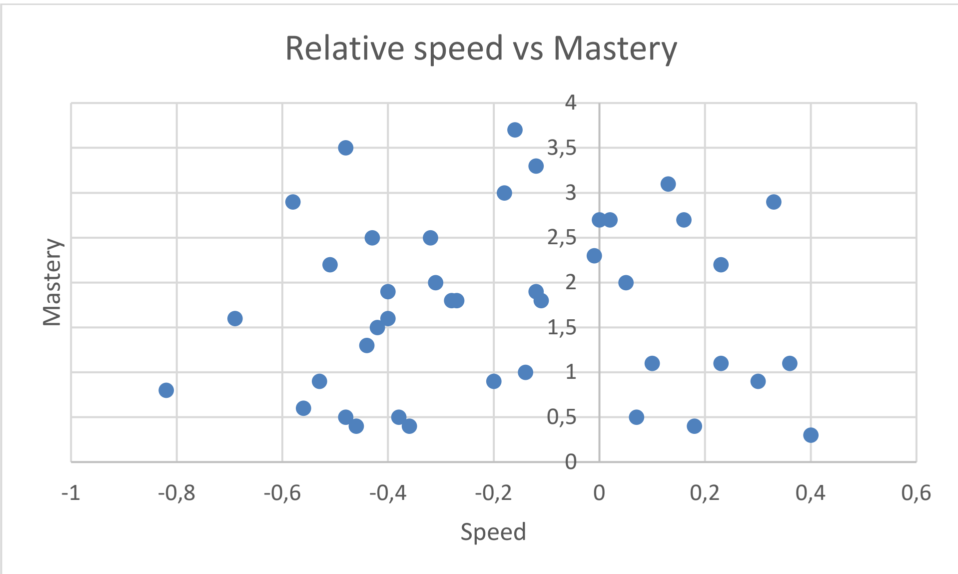


Figure 3 shows how fast and how accurately students were working. The speed is generated by the program relative to the entire student group and is referred to as relative speed. In this particular group of students the program capped the relative speed in the interval $[-2; 4]$. In the figure, a negative speed indicates that a student is working slowly relative to his/her peers while a positive speed indicates that a student is working faster in comparison with his peers. The normal relative speed value is calculated at 0 by the program.

The mastery levels ranges from 1 to 4 (lowest to highest). Earlier mastery levels were expressed as percentages so that the data read easily on Figure 2 since on the vertical axis of Figure 2 the number of exercises completed was big (0 to 500). It can be noted from Figure 3 that 28 out of 42 students were solving exercises slowly relative to their peers. Therefore, the 28 students have a negative speed. See also Figure 4 below:

Figure 4: Students relative speed and their mastery levels



Considering Figure 4, researchers assumed that some students were solving exercises slowly because they wanted to master a particular procedure since the mastery level of these students is higher compared to others. For example, the mastery level of Student

28 is 3,7 out of 4 while the relative speed of the same student is $-0{,}16$ relative to the whole group as indicated on the program. A possible explanation for this might be that Student 28 was putting more efforts and working slowly to ensure that he/she obtain correct answers and therefore improved mastery. Consequently, learning was taking place since the student was progressing through almost all the mastery levels covering exercises of low level difficulty to the highest level of difficulty.

On the other hand, student 40 has a mastery of 1,1 out of 4 and a positive speed of 0,23. Student 40 was working faster compared to Student 28 but with a low mastery level. The assumption is that Student 40 was obtaining inaccurate answers since he/she was working faster and therefore making more errors. Student 40 was putting less effort to study model solutions to achieve success in the next exercise. Therefore, the student did not acquire the desired mathematical procedure through the existing practice.

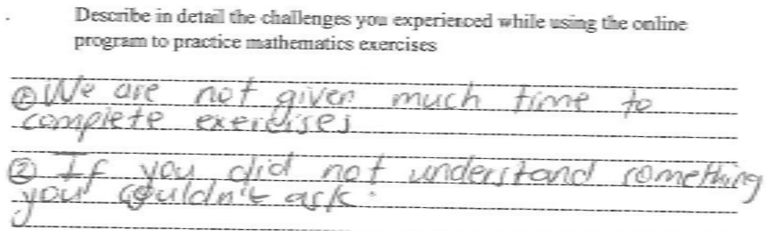
The mastery scores were averaged and the average mastery score is 1,73. The standard deviation of the mastery scores is 0,97. Therefore, the mastery level scores are not widely spread around the mean. The correlation coefficient is 0, 086 which means that there is a very small correlation between the relative speed and mastery levels.

DISCUSSION OF RESULTS

The findings of the study are discussed in the context of the empirical data gathered through students' reports and the questionnaire. As mentioned in the introduction, this study was set out to explore students' usage of an online interactive program designed to enhance mastery of mathematical procedures through repetitive practice of particular procedures. Researchers explores students' usage by considering the mastery score achieved by each student and the relative speed on practicing a mathematical procedure relating to simplification of exponential expressions.

The results of this study indicated that some students could have attempted exercises requiring skills with which they were already familiar and comfortable. Furthermore, scrutiny of students' program reports (number of exercises attempted) suggested that some students attempted the same type of exercise most frequently. The students' reports suggested that in the given time of two weeks available to students to complete 100 exercises, many students had engaged only with exercises requiring simple mathematical procedures considering their level of study. A possible explanation for this might be that students had insufficient time to complete exercises of next difficulty level or they did not understand the question clearly. From the open-ended questionnaire some students stated the following:

> "we are not given much time to complete activities". "If you did not understand something you couldn't ask".



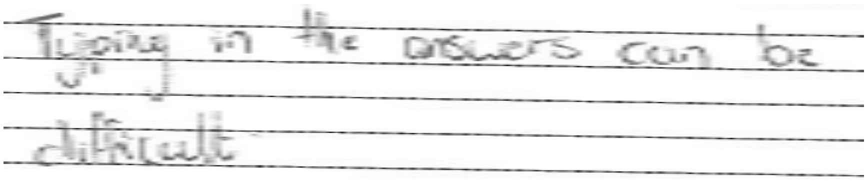
Describe in detail the challenges you experienced while using the online program to practice mathematics exercises

① We are not given much time to complete exercises

② If you did not understand something you couldn't ask.

This comment could indicates that the type of support provided by the program is insufficient to help this student. This student needed to ask a tutor or lecturer for assistance rather than solely rely on the program. It is also probable that some students never permitted the program to generate other types of exercises for further practice to develop the desired mastery. Among the 42 selected students, 7 students wrote a similar comment. In general, it appears that students put less effort on the practice of mathematical procedures. Consequently, their areas of weaknesses remained unaddressed in earlier exercises, and since subsequent exercises were likely to contain some aspects which require more attention. Choosing to use tools with which one is already familiar and comfortable is deemed an appropriate strategy (Foster, 2018). However, if the objective was central to why the teacher selected the task, then the task has failed automatically. Success on the program depends on obtaining progressive mastery levels. Students must show that they are progressing from level 1 to 4.

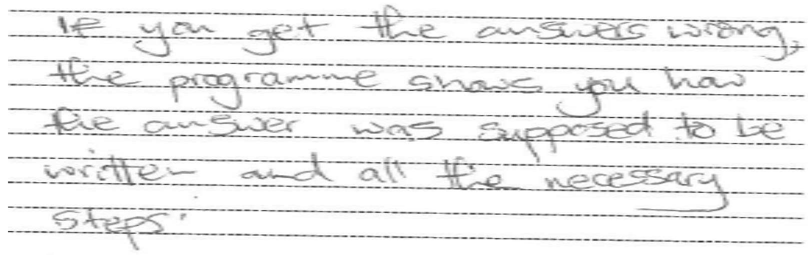
Some students, 11 out of 42 students, indicated that they were unable to type answers:



Typing in the answers can be difficult

Other students indicated that they benefited from using the online program:

> "if you get the answers wrong, the programme shows you how the answer was supposed to be written and all the necessary steps"



If you get the answers wrong, the programme shows you how the answer was supposed to be written and all the necessary steps.

Implication for future study is whether online interactive mathematics programs could be a viable alternative to traditional exercise practice, since the programs can offer possibilities of unlimited exercises.

TRUSTWORTHINESS AND CREDIBILITY

It was the researchers' responsibility to present as sound and impartial a report as possible. To fulfill this responsibility, the researchers considered the notion of trustworthiness and credibility. Trustworthiness and credibility are important keys to research that makes use of qualitative data. Poggenpoel (1998) identifies truth value, applicability, consistency and neutrality as four important criteria applicable to the assessment of credibility and trustworthiness. The researchers applied these four criteria to assess the trustworthiness and credibility of the findings. The researchers' position as a facilitator of the teachers participating in this study was not allowed to compromise the objectivity of his role within this study. Classroom observations were used to increase the trustworthiness and credibility of the study, as the researcher was physically present in the classroom.

ETHICAL CONSIDERATIONS

Ethics requires researchers to engage in moral deliberation within the context of research. This moral deliberation implies, amongst others, obtaining permission from parties involved in the study. In the case of this study, permission was obtained from the institution where the students participating in the study are enrolled. The students were informed that their participation is important, but that their role is voluntary and that they are free to withdraw should they feel uncomfortable during the course of the study. For purposes of confidentiality and anonymity, the identities of the students were concealed by using alphabetical letters as pseudonyms.

CONCLUSION

The aim of this study was to explore extended first-year extended engineering students' usage of an online interactive program designed to enhance mastery of mathematical procedures. However, the results of this study were not very encouraging. The outcome of this study suggests that the online program was not entirely effective for some students. The study indicated that 22 out of 42 students (almost half the number of the students in the study) did not master the necessary mathematical procedures required to simplify exponential expressions. The mastery level of the 22 students was lower than the average mastery score with regard to the simplification of exponents. Some students continued to type answers to exercises without showing an improvement in mastery levels. For such students, learning was not taking place since students did not manage to progress through all the difficulty levels relating to simplification of exponential expressions.

For the teacher whose objective is to develop mastery of mathematical procedures in students, the findings from this study show that the objective might be achieved through online exercise practice for certain students. Further studies using online interactive programs in other mathematical areas and with students outside this range would be necessary to extend the generalizability of these findings.

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THE PROCESS OF THE DEVELOPMENT OF A TOOLKIT TO SUPPORT PRE-SERVICE TEACHERS' UNDERSTANDING AND PRESENTATION OF LENGTH IN MATHEMATICS CLASSROOM

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South African learners perform poorly in mathematics, throughout all the school phases. The training of teachers, especially in the primary school level, has been identified as one of the contributing factors. When pre-service teachers leave training institutions, they are not adequately prepared to teach the mathematical content in the grades for which they have been trained. Added to this is the challenge of insufficient or inadequate material for them to competently teach various topics in the mathematics curriculum. The discrepancy in the material used by South African teacher training institutions to prepare the primary school teachers exacerbates the challenge and calls for the development of relevant material to facilitate the teaching of mathematics with understanding. This paper reports on the process of developing a Toolkit for teaching length to Intermediate Phase learners. Recommendations are also made on possible steps to follow, for other teachers to develop similar Toolkits.

INTRODUCTION AND BACKGROUND

The South African education system has been faced with many challenges, particularly in the context of the teaching and learning of mathematics. For example, teachers do not have sufficient mathematical content knowledge for the level of schooling they are supposed to teach (Bernstein, 2013; Spaull, 2013; Luneta, 2014). Also, teachers use procedural methods of teaching that do not promote conceptual and connected mathematical understanding (Graven & Venkat, 2017). Other challenges involve the shortage of relevant teaching and learning resources (Modisaotsile, 2012) for developing learners' understanding of mathematical concepts.

The training and development of teachers in Initial Teacher Education (ITE) programmes in South Africa has been identified as one of the areas that contribute to some of the challenges mentioned above. For example, despite the guidelines stipulated by the Department of Higher Education and Training (DHET) regarding the teaching and learning of mathematics, different universities differ in the way they interpret and implement these guidelines (Fonseca, Maseko & Roberts, 2018). Furthermore, mathematics courses offered to pre-service teachers (PST's) at different universities in South Africa vary in terms of the mathematics content dealt with. Some universities offer courses with advanced mathematics content, others empower PST's with mathematics at the grade level in which they will teach, while some focus on the mathematical thinking that teachers need to do (Ball, Hill & Bass, 2005).

In an effort to address the disparity between the ITE programmes in South African universities, the DHET then initiated a four year (2017 – 2020) Primary Teacher Education (PrimTEd) project, as part of its Teaching and Learning Development Capacity Improvement Programme (TLDCIP). The initial work done through the project culminated in the formation of several working groups (WG's), each dealing with a different focus area, such as Literacy/Reading, Number sense, Geometry and Measurement, Mathematical thinking, Knowledge management, Assessment and Work-Integrated Learning (WIL).

According to the PrimTEd project plan, four deliverables, namely, teaching standards curriculum frameworks, materials and assessment tools, need to be produced for the ultimate implementation by all South African universities (PrimTEd Newsletter, October, 2016). The Geometry and Measurement Working Group decided upon the approach of developing resources in the form of Toolkits for teaching various content topics in the Intermediate Phase (IP). This paper reports on the development of the Toolkits in the content area of Length.

The main research question for this study is:

1. What are the provisions in the CAPS document to help with the teaching and learning of the concept of length?
2. What materials are needed to support the teaching and learning of length?
3. What are the processes in the development of material to support the teaching and learning of length?

Pre-service Mathematics teacher education in South Africa

Despite the fact that mathematics education is regarded as one of the national priorities in South Africa, learners continue to be offered inferior mathematics education (Stols, et. al., 2015). The resulting poor performance of South African learners in several Trends in International Mathematics and Science Studies (TIMSS's) is evidence that there is a crisis in South African mathematics education (Spaull, 2013). Whereas the quality of teaching depends on the teachers' knowledge that they bring to the classroom (Ball, Hill & Bass, 2005; Charalambous, 2011; Rowland & Ruthven, 2011), teacher training in mathematics seems to be inadequate in that teachers lack mathematical content knowledge (MCK) as well as the skills required to present effective lessons. Consequently, learners suffer, because they tend to passively rely on the teacher for their success in mathematics, particularly, South African learners, instead of their own independent thinking and sense making of mathematics (Graven, Hewana & Stott, 2013). Therefore, if PST's are to be prepared adequately, teacher training should not only develop their MCK, but their content knowledge for teaching (CKT – M).

According to Hill *et. al.* (2008, p. 431), mathematical knowledge for teaching involves more than just the knowledge of the subject matter that is required to teach the subject, but also “the subject matter knowledge that supports that teaching. For example, why and how specific mathematical procedures work, how best to define a mathematical term for a particular grade level, and the types of errors students are likely to make with particular content”. Pre-service teacher education should therefore prepare and produce PST’s who are empowered with both the knowledge of concepts to be taught, as well as methods on how to implement processes that will result in quality instruction in their mathematics classrooms. The development of the IP Toolkit on length, for use by PST’s during their WIL sessions and beyond, is one way of providing PST’s with knowledge and skills to enable them to effectively deliver the mathematics curriculum in their classes.

Length as an attribute of measurement

One of the real-world applications of mathematics is measurement, which links two components of mathematics, namely, geometry and real numbers (Clements, 1999). Measurement involves assigning a number to indicate a comparison between the attribute of an object being measured and the same attribute of a given unit of measure (Van de Walle, Karp & Bay-Williams, 2015).

Length, as one of the attributes of measurement, involves measuring the distance between two points or the straight line between them (Sarama & Clements, 2009; Suggate, Davis & Gouding, 2010). Several big ideas or foundational concepts that underlie linear measurement have been identified. These include unit iteration, accumulation of distance, conservation, additivity, equal partitioning, transitivity and relation between number and measurement (Sarama & Clements, 2009; Van de Walle *et. al.*, 2015; Tan-Sisman & Aksu, 2016; Hansen, 2017). These ideas form part of the Toolkit that we developed and will be discussed later in this paper.

The South African Curriculum and Assessment Policy Statement (CAPS) on mathematics in the Intermediate Phase stipulates the requirements for teaching the concept of length. The main concepts and skills that learners in the IP need to master are: estimation and practical measurement of length using appropriate measuring instruments and formal units of length; recording, comparing and ordering lengths, in different contexts; calculations and problem-solving involving length, including conversions between different units of length (DBE, 2011a). The big ideas and key concepts involved in the measurement of length, as mentioned in the preceding paragraph, have to be applied during the teaching and learning of the length concepts as prescribed in the CAPS. However, most of these big ideas and concepts are not explicit in this curriculum document. Hence a Toolkit on length has to fill in these gaps by including such concepts.

PST’s need to be empowered with skills to address learners’ errors and misconceptions when dealing with the concept of length. Research shows that learners find the measurement of length challenging. For example, they tend to leave gaps between units or overlap units when they perform the iteration procedure (Lehrer, 2003). Other errors made by learners include the incorrect alignment of the measuring instrument with the object whose length is being measured (Tan-Sisman & Aksu, 2016). They also read the scale incorrectly by, for example, counting the number of marks on the instrument, instead of the number of spatial intervals in the distance spanned by length (Solomon & Vasilyeva, 2015). Errors and misconceptions in the measurement of length do not feature in the CAPS and this gap is identified and addressed in the Toolkit we developed.

The development of a Toolkit on teaching the topic of length in the IP

The Toolkits for teaching length in Grades 4, 5 and 6, were designed to cover, as a minimum, concepts and skills as stipulated in the South African CAPS for the IP. The aim of the Toolkits is to provide guidance with sets of materials in the form of content and pedagogical guidelines, which pre-service teachers can use in order to, firstly, understand themselves the key concepts and theory involved in the topic, and secondly, know how to teach the topic competently to IP learners. Therefore the Toolkits consist of CKT-M and MCK on length, as well as Pedagogical Content Knowledge (PCK) involved in the measurement of length. PCK "bridges content knowledge and the practice of teaching, ensuring that discussions of content are relevant to teaching and that discussions of teaching retain attention to content" (Ball, Thames & Phelps, 2008, p. 3).

The components of the Toolkit on length are shown in Table 1 below.

TABLE 1: The components of the Toolkit on length

Section A	**Section B**
1. Theory around the concept of length: • Definitions pertaining to the measurement of length. • Explanation of related terms and concepts. 2. CAPS requirements on the topic, for the three grades. 3. Big ideas and key concepts in the measurement of length. 4. Errors and misconceptions associated with the teaching and learning of length.	1. Focus areas of each sub-topic of length, as stipulated in CAPS. 2. Learners' prior knowledge, linked to each focus area. 3. An introduction for each sub-topic. 4. Activities to be done with learners to facilitate mastery of concepts and skills. 5. Formative assessment for each activity; possible solutions. 6. Summative assessment for each sub-topic; possible solutions. 7. Clarification notes and teaching guidelines. 8. List of resources used for each sub-topic.

Conceptually oriented and procedurally oriented tasks have been included purposefully in dealing with the topic of length in the Toolkits. Procedurally oriented tasks are routine tasks whereby previously learned step-by-step solution methods or mathematical computations are utilized, whereas conceptually oriented tasks involve the use of non-routine and original tasks that require understanding of fundamental principles or concepts (Tan-Sisman & Aksu, 2016). A gap seems to exist in the current DBE curriculum material in that conceptual understand of various sub-topics on length is not explicitly specified in the CAPS, nor is it adequately covered for most sub-topics in the DBE Workbooks. For example, in the Grade 4 Workbook, quite a number of questions and tasks on length are procedural in nature, with instructions such as: "Measure each object and give your answer in cm and mm. Order the objects from shortest to longest", or "Complete the numbers on the ruler, measure the lines and complete the table" (Department of Basic Education 2019, pp. 110 – 111). Both tasks in this case involve the procedure of aligning the measuring instrument with the object or line whose length is being measured, without much demand for conceptual understanding. Thus deep mathematical thinking is not encouraged.

In the Grade 4 Toolkit, a conceptually oriented task, comparable to the tasks given above, appears in Figure 1a below, with the following instruction:

> "Your friends, Thando and Lucy, have a disagreement about the height of the picture of the tree in the diagram below. Thando says that the picture has a height of 3 cm while Lucy says it has a height of 8 cm. Which one, if any, of your friends is correct? Explain your answer".

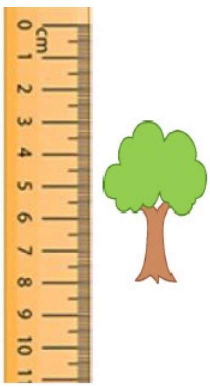


FIGURE 1a: Conceptually oriented task

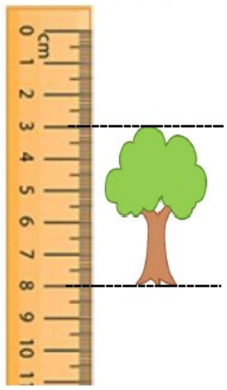


FIGURE 1b: Possible solution strategy

In the above task, learners are encouraged to think about why the height of 3 cm or 8 cm, or both might be wrong. In the process they intuitively use the fundamental principles of accumulation of distance and additivity. That is, they need to understand that "as you iterate a unit along the length of an object and count the iteration, the number words signify the space covered by all units counted up to that point" (Sarama & Clements 2009, p. 276). Therefore they would realize that the height of the picture does not span over 3 units (cm) or 8 units (cm). Hence they would have to find correct ways of determining the height, such as "to subtract the initial measurement from the final measurement" (DBE 2011a, p. 73). Alternatively, they would have to begin counting the iteration from any origin through the interval spanned by the height, up to the other endpoint. They might even have to think of drawing a line to make it easier for themselves to read the measurements, as depicted in Figure 1b above.

Another shortcoming with the DBE CAPS and Workbooks is the explicit incorporation of learners' possible errors and misconceptions in the teaching and learning of the concept of length. Yet errors and misconceptions can be used as catalysts for learning, whereby learners' mathematical understanding is developed through identifying and addressing their errors (Tan-Sisman & Aksu, 2016). Therefore it is suggested that errors and misconceptions should be included in the mathematics curriculum to inform teachers beforehand about learning challenges that they can expect (Ryan & Williams, 2007). Hence the Toolkits on length specify possible learners' errors and misconceptions that could be expected, as well as suggestions towards addressing them.

CRITICAL STEPS IN DEVELOPING A TOOLKIT ON LENGTH

Below we outline the steps taken and used as a guide in the development of the Toolkit. Examples of material from the Toolkit are also included to provide clarity.

Analysis of SA curriculum documents

The point of departure in the development of the Toolkit was the analysis of the SA curriculum requirements for teaching the topic of length in the applicable grades. The following documents were consulted: CAPS; *Guidelines for responding to learner diversity in the classroom through CAPS*; *Mathematics Teaching and Learning Framework for South Africa: Teaching mathematics for understanding*; the Workbooks and other Learning and Teaching Support Material (LTSM) for the various IP grades. The concepts, skills and processes involved in mastering the content involving length were identified and gaps were addressed when designing the activities to be done by learners.

The design of activities in each grade Toolkit was based on critical information found in the CAPS as depicted in the table below:

TABLE 2: CAPS information guiding the development of activities in the Toolkits

Grade 4	**Grade 5**	**Grade 6**
Term 2: 7 hours (DBE 2011a, pp. 73–75)	Term 2: 6 hours (DBE 2011a, pp. 163–165)	Term 3: 5 hours (DBE 2011a, pp. 272 – 274)
Activities were based on sub-topics for length that were, in turn, created based on the concepts and skills stipulated on the pages as indicated above. The number of sub-topics for each grade corresponds to the number of hours spent on the topic of length for that particular grade.		

The activities in the Toolkit are to be mediated through lesson plans, with the number of lesson plans corresponding to the number of hours allocated for each grade.

However, it is our belief that teachers should have the responsibility and opportunity to design their own lesson plans, based on the kind of learners they have in their classes. For example, they have to cater for differentiations according to the needs of their learners. However, sample lesson plans are included in the Toolkit as guidance.

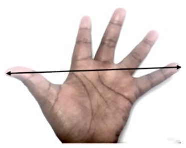
Other information from CAPS that features in the Toolkit relates to integration within mathematics, as well as across other subjects. For example, mental mathematics, which has to be incorporated into and done during each lesson when teaching length, belongs to the content area of Numbers, Operations and Relationships. CAPS lists the concepts and skills relating to mental mathematics for each sub-topic. Integration across other subjects is catered for in the Toolkit, with activities such as one on comparing and arranging lengths of different colour crayons. Suggestions are then given for further reinforcement when colours are taught during Life Skills lessons.

The *Guidelines for responding to learner diversity in the classroom through CAPS* document provides information to teachers on how to facilitate and support curriculum differentiation in the classroom. Analysis of this document reveals that teachers are expected to create and use flexibly their own innovative material, methods of presentation, learning activities, lesson organization and assessment (DBE, 2011b). The document outlines generic strategies from which teachers can base their own innovations. However, teachers lack mathematical content knowledge and skills needed to apply what they know in the classroom (Stols *et. al.*, 2015). Also, the training of PST's does not prepare them for rethinking their beliefs surrounding the teaching profession and adapting to new ways or strategies of mediating the content they teach (Lauwerier & Akkari, 2015).

The Toolkit on length provides PST's with specific strategies for responding to learner diversity in the context of each sub-topic presented. For example, learning material allows learners to engage with hands-on activities that provide for tactile and bodily-kinesthetic learning experiences (see Figure 2 below).

Introduction to sub-topic: Rounding off numbers in the context of length

Estimate the length of your hand span in centimetres. Remember that your hand span is the distance from the tip of your thumb across the palm, to the tip of your small finger, whey your hand is opened as wide as possible. Refer to the picture below:



Now use a ruler to measure your hand span in centimetres.

FIGURE 2: Activities involving tactile and bodily-kinesthetic learning experiences

Electronic material, in the form of Geogebra, is used in the Toolkit, to further respond to learner diversity, in addition to the use of geoboards, which is suggested in CAPS (DBE 2011a, p. 232).

The document on guidelines for the promotion of learner diversity stipulates that learning activities should include problem solving that allows learners to explore concepts rather than reproduction of prior knowledge (DBE, 2011b). The Toolkit on length provides specific activities for PST's to facilitate such problem solving activities with learners (see example in Figure 3 below).

Introduction to sub-topic: Measuring lengths practically and accurately, using a ruler

The class teacher gives new pencils to learners as soon as the length of the pencil becomes less than half of the original length of a new pencil. She does this so that learners do not struggle holding the pencil when using it. Let us look at Sino's and Angel's pencils. How can we find out if the teacher needs to replace the two learners' pencils or not?

This is a new pencil: 

This is Sino's pencil: 

This is Angel's pencil: 

FIGURE 3: An activity that promotes problem-solving

The purpose of the document on *Mathematics Teaching and Learning Framework for South Africa: Teaching mathematics for understanding* is "to attend to the challenges associated with the teaching and learning of mathematics, so that learner outcomes are improved" (DBE 2018, p. 3). Analysis of this document reveals that the framework is mainly embedded in the theory of mathematical proficiency, which proposes five strands for promoting mathematics understanding, namely, conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition (Kilpatrick *et. al.*, 2001). Only the first four of the strands (or dimensions) are applicable in the framework. Examples of how to apply the four dimensions are provided for each phase of schooling. Teachers are then left to design their own learning centred activities and methods. However, most teachers are not able to provide opportunities for learner-centred instruction (Lauwerier & Akkari, 2015), and they give little encouragement for learners to "predict, describe, justify, represent or construct mathematical knowledge" (Luneta 2015, p. 5).

In the Toolkit on length, suggestions are given for each activity on how to develop learners' mathematical understanding. The example below was designed to develop learners' conceptual understanding, procedural fluency and adaptive reasoning involving the sub-topic: Converting between different units of length – mm, cm and m.

During athletic sports day, learners participated in long jump. The lengths of their jumps were measured in metres, using a tape measure. The top (longest) two distances jumped belonged to Sethu and Ayanda, which were both recorded as 3 cm. The two athletes had something to say about this score:

1. Ayanda claimed that there is something wrong with this measurement. The athletics officials examined this claim and realized that they made an error. What error could this be?
2. After the error was corrected, Sethu argued that the scores could have been different if smaller units were used when taking the measurement. What units could those be and how could this have possibly made a difference in the recorded measurements of the distances?

Analysis of other Teaching and Learning Support Material (TLSM)

The South African DBE prescribes and recommends certain material to support the delivery of the curriculum. These are mainly the Workbooks for the different IP grades and some textbooks from various authors and publishers. An analysis of these books reveals that they do not fully support the development of learners' mathematical proficiency as discussed in the preceding section. For example, the instructions from the grade 4 Workbook, as seen in the examples given in an earlier section of this paper, and many other exercises in the Workbooks, do not consciously encourage learners to apply some strategic competence or adaptive reasoning. They mostly promote procedural fluency and some conceptual understanding.

Other textbooks, which are recommended by the DBE, do give considerable guidance for teaching concepts and skills on length in order to teach for mathematical understanding, often in the Teacher's Guide (Human *et. al.*, 2016). However, they do not provide much theoretical underpinning of such concepts and skills to empower PST's with deep mathematical knowledge (of the concept of length in this case). The Toolkit then closes this gap by including theoretical dimensions to the teaching and learning of length, such as the key concepts and big ideas that form the foundation for children's understanding of linear measurement (see Table 3 below). These key concepts are not sufficiently supported by traditional measurement instruction (Sarama & Clements, 2009).

TABLE 3: Key concepts in linear measurement (Sarama & Clements 2009, pp. 275 – 277)

Concept	Description
1. Understanding of the attribute	Lengths span fixed distances
2. Conservation	The understanding that as an object is moved, its length does not change.
3. Transitivity	The understanding that if the length of object X is equal to (or greater/less than) the length of object Y and object Y is the same length as (or greater/less than) object Z, then object X is the same length as (or greater/less than) object Z.
4. Equal partitioning	The mental activity of slicing up an object into same-sized length units.
5. Unit iteration	The length measure is constituted through iterating the same unit along the length of the object being measured, without gaps or overlaps.
6. Accumulation of distance and additivity	The understanding that as you iterate a unit along the length of an object and count the iteration, the number words signify the space covered by all units counted up to that point.
7. Origin	The notion that any point on a ratio scale can be used as the origin.
8. Relation between number and measurement	Understanding that the units of length measurement being ‘counted’ are continuous rather than discrete; the number is used to quantify this distance in real-world contexts; there is an inverse relationship between the size of the unit and the number of units in a given length measure.

Analysis of Mathematics Education curriculum for PST’s

Some shortcomings of the pre-service mathematics teacher education in South Africa have been discussed in the section on the background to this study. In this section we highlight the inadequacy of the PST Mathematics Education curriculum to prepare PST’s for the development of their own material. Hence the need to design Toolkits that will be used by the PST’s and assist them in facilitating learners’ mathematical understanding.

When they enter the teaching career, South African teachers do not have the necessary mathematics subject content knowledge (Van der Sandt, 2007). The poor mathematical background tends to make them rely on the textbooks and other materials, which they have not developed or adapted themselves (Feza & Webb, 2005). However, textbooks tend to present or explain mathematical ideas and concepts in simplistic and routine-like ways without, for example, providing for multiple strategies to solutions (Luneta, 2015). The Toolkit on length, in contrast, has been consciously designed to provide opportunities for the development of strategic competence as one of the strands of mathematical proficiency.

CONCLUSION AND RECOMMENDATIONS

The South African CAPS for IP provides clear guidelines on the knowledge and skills that need to be developed in learners when teaching the concept of length. However, it lacks clarity on the big ideas and key concepts that teachers need to, firstly understand themselves, and then develop in the learners to which they teach measurement of length. The CAPS also does not adequately empower teachers with the kind of errors and misconceptions they can expect when teaching length to their learners. Other curriculum documents that are meant to support the CAPS also stipulate the outcomes which the teaching of mathematics, and length in the context of this paper, should achieve. These documents provide generic statements and expect teachers to use these to design their own learning experiences for developing learners' mathematical understanding. These expectations present a challenge, given that teachers and PST's are not adequately empowered with skills for designing their own material. It is also evident that available LTSM on teaching the topic of length to IP learners does not fully promote the development of mathematical proficiency in learners. Additional LTSM that is available to teachers and PST's also has some shortcomings in terms of providing them with deep mathematical knowledge that is required to teach the concept competently.

The development of the Toolkit on length is meant to bridge the gaps in the currently available material and provide adequate support for teaching the topic. The current Toolkit is a working document that still needs to be tested with B.Ed. IP PST's to determine the extent to which it develops their understanding of length and its theoretical underpinnings, as well as their pedagogical knowledge to allow them to mediate the content to learners. This paper has outlined the processes and steps involved in the development of the Toolkit on length. We recommend the steps, as summarized in Table 4 below, and hope that they would help PST's and other teachers to develop similar Toolkits.

TABLE 4: Steps involved in the development of a teaching and learning Toolkit

1. Analysis of curriculum documents	Identify the knowledge, concepts and skills that need to be developed regarding the topic of concern, from the main curriculum document. Determine and note any lack of clarity or omissions regarding the concepts and skills or how to facilitate learners' development of these. Check if there are other additional curriculum documents that support the main document; Identify their purpose, focus, expected outcomes, their relevance to the topic concerned and any shortcomings.
2. Analysis of other available LTSM	Explore several LTSM and determine the extent to which they facilitate the development of the concepts and skills stipulated in the curriculum documents; note any shortcomings.
3. Analysis of Mathematics Education curriculum for PST's.	Find literature that documents Mathematics Education curriculum for the PST's to determine if it is adequate in preparing them for mediating the teaching and learning of the concepts and skills applicable to the topic, given the available resources. Include theoretical aspects of the topic.
4. Development of the Toolkit	Using all of the above as guidance, develop the Toolkit, and address any shortcomings identified; this becomes the working document that will need to be evaluated, modified and strengthened as the case might be.

The above steps are not cast in stone; they provide for flexibility in application and are meant to provide a possible pathway to follow when developing similar Toolkits.

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MATHEMATICAL LITERACY, MATHEMATICS IN CONTEXT: WHOSE CONTEXT IS THIS FOR?

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This paper presents arguments on the issue of contexts in mathematics with a special reference to mathematical literacy. Central to the discussion, on one hand, is the positive role of contexts in mathematical literacy to make mathematics accessible; on the other hand, is the dilemma and conflict brought by the context if it does not relate the real life of the learner. Findings and discussions reported in this paper are part of the bigger study which investigated teachers' interpretations and implementations of the intended mathematical literacy curriculum. In the main study, three forms of data collection instruments were used to gather data, namely, questionnaires, interviews, and observations. For the purpose of this paper, only findings from the classroom observations are presented and discussed. The conflict between real life contexts used in the lessons and learners experiences is discussed as emerged from the lesson observations.

INTRODUCTION

Mathematical Literacy was introduced as a new subject in Grade 10 in 2006, in Grade 11, 2007 and in Grade 12 in 2008. According to the DoE (2003), the purpose of the inclusion of Mathematical Literacy as a compulsory alternative to Mathematics, as a subject in the FET curriculum is that it will ensure that South African citizens of the future become highly numerate users of mathematical skills. The inclusion of contexts in Mathematical Literacy can be seen as one way of crossing the boundaries between mathematical and non-mathematical discourses; thus extending more opportunities to everyone to access mathematics. This can be explained, in terms of Bernstein (1996), as weak classification. Mathematical Literacy is considered to be weakly classified. There is substantial evidence in the literature, and from the findings of this study, that the manner in which Mathematical Literacy curriculum is designed and presented attempts to (i) draw the interest of the learners to do Mathematical Literacy without any fear of the mathematics content attached to it; (ii) to access mathematics; and (iii) to understand mathematics in real life situations. The aim of this paper is to discuss the issue of contexts used in mathematical literacy as the central aspect of mathematical literacy. The paper is guided by a critical question which: Whose context is presented in mathematical literacy?

LITERATURE REVIEW

The literature as presented below suggest that there are many views on the role of context/s in mathematics and science education. Some teachers believe that contexts play a major (positive) role in the teaching and learning of Mathematical Literacy. Researchers argue that contexts in mathematics can play a major role in making mathematics accessible (see: Blinko, 2004; Boeler, 1993; Mudaly, 2004; Van Den Heuvel-Panhuizen, 2005; Nicol and Crespo, 2005). Similarly, in mathematical literacy, contexts play a role in making the mathematics accessible.

The influence of contexts in arousing learners' interest has been strongly argued in the literature (see Boaler, 1993; Bowman, 1997; Mudaly, 2004; Blinko, 2004; Nicol and Crespo, 2005). Mudaly (2004) argues that besides the idea of showing learners how mathematics is related to the real world, contexts also serve to increase interest in the subject matter. Drawing from his experience, Browman (1997) asserts that after allowing his students to work with real-world problems in his class, the level of student interest increased to the extent that they were especially excited about being able to solve a mathematics problem that even the so-called mathematics geniuses in calculus could not solve. In this regard, Blinko (2004) maintains that "putting [mathematics] questions into a context can go a long way in making abstract ideas more meaningful" (p.3.). According to Blinko (2004), contexts make mathematics meaningful to the learners. Boaler (1993) adds that using real world, local community, and even individualized examples which students may analyze and interpret, is thought to present mathematics as a means with which to understand reality.

Similarly, Nicol and Crespo (2005) emphasize that "the contexts in which mathematics is studied play an important role in helping students understand not only how, when, and why particular concepts, procedures, and skills are used, but also what makes them significant and worth knowing" (p.240).

The results in this study show that in the two selected lessons for this paper, the context was not seen enabling the learners to access mathematics as depicted in Blinko (2004); Boeler (1993); Mudaly (2004); Van Den Heuvel-Panhuizen, 2005Nicol and Crespo, 2005. This leads to an important question as to what type of context should be used to enable learners to access mathematics?

Apart from the positive impact of the contexts in mathematical literacy, studies by different researchers in South Africa and internationally, (see, Cooper and Dunne, 1999; Chacko, 2004; Naidoo and Parker, 2005; Murray, 2003; Greer, 1993; Verschafel and De Corte, 1997) have shown that context can sometimes affect understanding. Some of the negative experiences reported by teachers in this study relate to the language of learning and teaching. The argument from the teachers is that mathematical literacy has a variety of contexts which demand language proficiency or competence from the learners. Teachers contend that most of the learners have a problem understanding English and hence find it difficult to interpret Mathematical Literacy problems because of the language issue. This argument is supported by the qualitative study that was conducted in Korea (Whang, 1999), and which revealed that children have difficulties in solving math word problems written in English. Similarly, a study of Mathematics Literacy of final year students (Howie and Pietsern, 2001) showed that students performed particularly poorly in questions requiring written answers:

Students showed a lack of understanding of mathematics literacy questions, and an inability to communicate their answers in instances where they did understand the question (p.19). Murray (2003) argues that for a child to understand and respond to a problem posed, the language and grammatical constructions used when the word problem is formulated are obviously crucially important (p.39).
Apart from linguistic demands that contexts bring into the teaching and learning of Mathematical Literacy, in particular, research has shown that contextualizing mathematics can sometimes produce undesirable results (see: Cooper and Dunne, 1999; Chacko, 2004; Naidoo and Parker, 2005; Murray, 2003; Greer, 1993; Verschafel and De Corte, 1997). The study by Cooper and Dunne (1999) has revealed how contextualizing mathematics creates some difficulties for working-class students, such that they perform significantly more poorly than their middle-class peers on contextualized tasks, while their performance on decontextualized tasks is equivalent. The study on the implications of mathematics teachers' and officials' identities to mathematics discourses for democratic access to mathematics (Naidoo and Parker, 2005) involved seven Grade 9 teachers. All seven teachers expressed a negative orientation towards contextual mathematics. Some of the teachers maintained that:

> Teaching and assessing mathematics from situations denies pupils adequate subject content and knowledge (p.63).

Murray (2003) argues that the inclusion of context in mathematics does not necessarily produce good results all the time. Murray asserts that learners experience real life very differently from adults, and are familiar with very different aspects of real life. She stipulates four ways in which the context can act as a barrier to understanding mathematics: (i) Learners are not familiar with the context, (ii) The context has unpleasant connotations, (iii)The context is limited and (iv) The problem has to be transformed or modelled by the learner before he/she can solve it (p.40). Similarly, Van Den Heuvel-Panhuizen (2005) suggests four important points about context as a barrier to teaching and learning mathematics (i) context can hinder finding an answer (ii) students' unwillingness to take into account the context (iii) context problems do not allow one to take the context into account and (iv) taking the context into account is not evenly distributed among students.

THEORETICAL FRAMEWORK

For the theoretical framework, I have drawn from Basil Bernstein's (1975; 1982; 1996) framework of knowledge system and the Third International Mathematics and Science Study (TIMSS) (1996) framework of curriculum analysis perspective to analyze the Mathematical Literacy curriculum and the teachers' interpretations of this curriculum, with special reference to the intended and implemented the curriculum. Bernstein (1971) contends that there are three message systems through which the formal education knowledge can be recognized, that is, the curriculum (defines what counts as valid knowledge), pedagogy (defines what counts as a valid transmission of knowledge) and evaluation (defines what counts as valid realization of this knowledge on the part of the taught). In this study teachers' interpretations and implementation of the curriculum were viewed through the lense of what counts as a valid knowledge and a valid transmission of knowledge.

The TIMSS framework

The TIMSS (1996) framework for curriculum analysis has been used to analyze both curricula in South Africa (see: Taylor and Vinjevold, 1999; Taylor et al., 2003) and internationally (Robitaille, Schmidt, McKnight, Valverde, Houang and Wiley, 1997; Valverde, Bianchi, Wolfe, Schmidt and Houang, 2002; and Johansson, 2005). The TIMSS (1996) framework is based on a model of curriculum that has three components: the intended curriculum, the implemented curriculum and the attained/achieved curriculum. For the purpose of this study the third component (attained curriculum) will not be explored in great detail, as it is not within the scope of this study.

These key components are summarized in the following figure as detailed below:

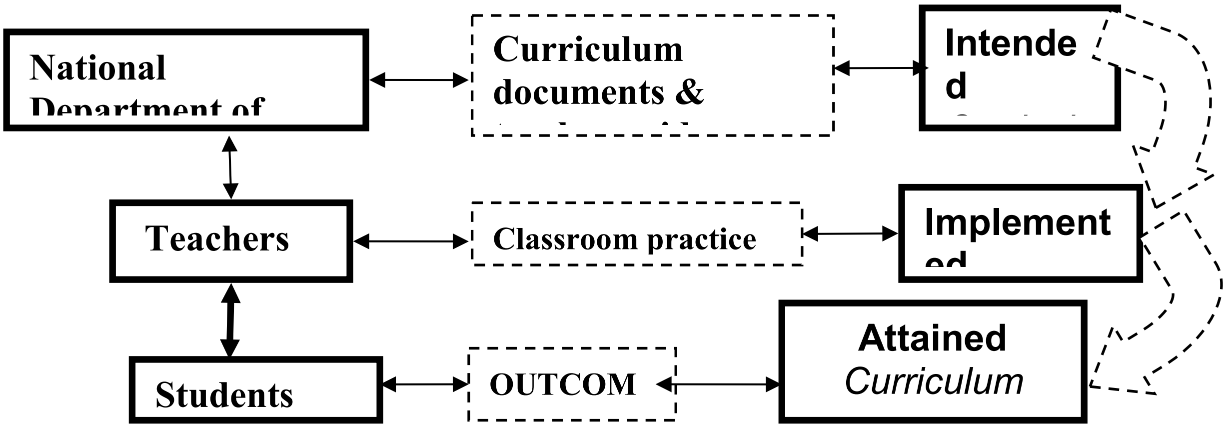


Figure 2: TIMSS Framework (TIMSS 1996)

The intended curriculum

According to TIMSS (1996), the intended curriculum consists of the mathematics and science that society intends students to learn, and the education system that society believes is best designed to facilitate such learning. Cuban (1995) refers to the intended curriculum as the official curriculum, and describes it as what state and district officials set forth in curricular frameworks and courses of study.). In the South African education system, the intended curriculum is designed by the National Department of Education and, after necessary consultations with all stakeholders, it is approved and adopted in parliament as a policy (for example CAPS Mathematical Literacy Grades 10 – 12 (general)).

The implemented curriculum

According to TIMSS (1996) the implemented curriculum is made up of what is actually taught in the classroom, who teaches the curriculum, and how it is taught. Robitaille et al. (1997) contend that intentions and objectives at the level of teacher and classroom activity are considered as the implemented curriculum.

RESEARCH METHODOLOGY

As indicated in the abstract, the main study used questionnaires, interviews and classroom observations to collect data. In this paper, I report on findings from classroom observations. The qualitative research approach was adopted for its relevance to this study. Two experienced Mathematical Literacy teachers (Mr. Khumalo and Mr. Alfred) were observed teaching Mathematical Literacy in grade 11 classes. Four consecutive lessons were observed with each participant, followed by teacher reflection on each lesson. The two lessons from each participant were selected on the bases of representative approaches used by the individual teachers.

FINDINGS AND DISCUSSION

Findings of this research particularly on the two selected lessons which I found presenting conflicting views on the role of the contexts used in Mathematical Literacy lessons, particularly when the context used does not relate to the real-life experiences of the learners.

Khumalo' s Lesson.

After introducing the probability scale, Khumalo introduced the percentage (%) on the scale that 0 represents 0%, 0.5 represents 50% and 1 represents 100%. He then gave them a class exercise taken from the textbook. It was noted that there were only four of these textbooks available in a class of 24 learners. The learners were required to share books and work on the class exercise as a group. See the exercise below:

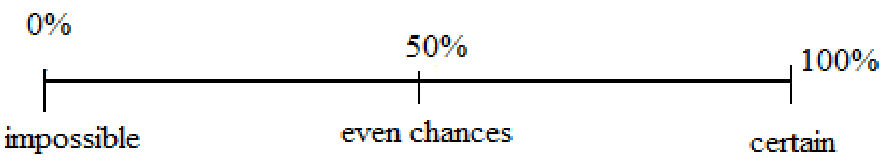
Class activity 1: Probability scale

Copy the percentage probability line: Fill in probabilities for the following on the number line:

Class activity 1: Probability scale

Copy the percentage probability line: Fill in probabilities for the following on the number line:

(a) It will get dark tonight
(b) It will rain this month
(c) This year there will be a drought
(d) A coin will fall on heads when tossed
(e) It will snow in Polokwane



Source*: Understanding Mathematical Literacy (*p.255.)

In this lesson, a variety of contexts were used. In the introduction, the blue and red balls were used to explore the concept of probability. It was noted that in number (e) above, the learners were confused by the context used. Learners did not know where Polokwane was, and they did not have any idea of the weather patterns in Polokwane. There was also confusion about the question (c) if there will be 'a drought'. Further confusion was caused by the time of the year in which this exercise was done, i.e. October. In this particular lesson, it is clear that the context used was not related to the learners' real life thus brought confusion amongst the learners.

Alfred's Lessons: Lesson 1 – Business Mathematics
The teacher introduced the lesson by reviewing key concepts dealt with in the previous lesson. The teacher then explained the concepts of fixed costs and variable costs. He then gave learners a class activity from a worksheet that was distributed to all learners. He requested a volunteer to lead class discussion. He chose a boy who was showing interest and willingness to lead the discussion.
The task was, as shown below, about the monthly expenses for Poncho's Portable Phones.

Class activity 1

Poncho's Portable Phones: income and expenditure

Item	***Price***
Rent	*R5 100,00*
Salaries	*3 x R 3 4 20, 00*
Coffee and tea	*R500, 00*
Stationery	*R975, 00*
Staff clothing	*2 x R156, 00*
Petrol	*R431, 72*
Cleaning supplies	*R87, 23*
Telephone	*R622, 97*

a) Determine which expenses are fixed and which are variable.

b) Calculate the total amount that Poncho's Portable Phones spends on business expenses.

The learner leading the discussion led the class in answering questions 1 (a) and (b). The teacher took a back seat and allowed the learners to discuss the answers. The lead learner was strong and confident and was able to control the learners, even those who were out of order. He did not accept any answer – unless the learner who was giving the answer gave reasons. There was a long and interesting argument about the price for ‘telephone’, as to whether it is a fixed or a variable cost. Some groups were saying ‘the rent for the telephone is fixed’ while others were saying ‘the telephone bills vary from month to month, depending on how much you have used the phone’. The teacher did not enter into the discussion until they had completed Activity 1.

These two examples given above show that a problem with contexts is not only limited to language per se, but also to other variables that come into play in teaching and learning Mathematical Literacy.

This study illuminates connections and disconnections (coherence and departure) between the intended curriculum and the implemented curriculum, and furthermore shows that teachers’ interpretations and recontextualization of the intended curriculum in classroom contexts are key to the nature of the curriculum that is implemented. While teachers endeavoured to ensure valid transmission of knowledge, on the side of the leaners there was no realization of valid knowledge due to the contexts used in the lesson.

CONCLUSION

This paper presented conflicting views on the role of context in mathematics as shown in the literature review section. It also confirms through empirical data from the two lessons observed that indeed contexts used in the teaching of Mathematical Literacy do not always enable learners to access mathematics. The contexts’ used in classroom lessons should be carefully selected so that they do not hamper learning on the part of the learners. It is recommended that teachers should be very careful about using examples or exercises taken directly from a textbook. Contexts used in textbooks cannot address every classroom situation; some may be too localized, while others may be too universal. Therefore, teachers should be very careful in selecting tasks and engage learners using context to develop and promote mathematical thinking and problem-solving.

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AN ANALYSIS OF PRE-SERVICE MATHEMATICS SECONDARY TEACHERS' COMMON CONTENT KNOWLEDGE OF PARABOLA AND EXPONENTIAL FUNCTIONS

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The tertiary training of pre-service mathematics teachers is pivotal in their professional preparation and formation as qualified educators. Multiple authors posit that pre-service teachers require a development of common content knowledge and specialised content knowledge not only of the topics they learn at university but of the topics, they will teach at school. The findings reported here is one of the series of papers exploring pre-service teachers' common content knowledge and specialised content knowledge of functions. Data was generated by means of activity sheet which was administered to second year mathematics students (n=179) in three tertiary institutions in the province of KwaZulu-Natal. It was troubling to discover that majority of students had lot of conceptual and procedural errors. Based on the findings, we concluded that as pre-service teachers' progress with mathematics content they learn at university, common content knowledge of parabola and exponential function, which is part of the concept they will teach, is weakening.

INTRODUCTION

There is no doubt that mathematics teachers need to have the expertise to identify and eliminate mistakes made by learners in mathematical concepts. However, for teachers to be able to do so they need to have a sound common content knowledge of the concept themselves. It is assumed that when teachers have graduated with their Bachelor degree they have acquired the necessary knowledge to teach the concepts, however, several studies have revealed that mathematics teachers lack the necessary knowledge needed not only to teach the subject but to solve the problems they expect their learners to solve (Bansilal, Brijlall, Mkhwanazi, 2014, Pournara, 2016; Adler 2017). There has been a growing body of literature concerning the preparation of pre-service mathematics teachers. Ball, Phelps and Thames (2008) have explicitly specified that development of pre-service mathematics teachers' subject matter knowledge of school mathematics concepts is necessary at the exit level in order for such teachers to become master-craftsperson in the teaching of mathematics. Drawing from Ball's model of mathematical knowledge, subject matter knowledge is divided into two strands of mathematical knowledge, i.e. common content knowledge and specialised content knowledge. This paper is part of a series of papers reporting on pre-service teachers' subject matter knowledge of functions. While research on learners understanding of mathematics concepts suggest that topics such as geometry and trigonometry are the ones learners find difficult to understand, research on the teaching of mathematics suggest that there is a challenge in the teaching of all mathematical concepts taught in schools. The Department of Education diagnostic reports further show that there is a challenge in the teaching of all mathematical concepts. We therefore opted to explore pre-service mathematics teachers' subject matter knowledge of functions because knowledge of functions is applicable to many mathematical concepts. However, in this article, we report on the findings based on their common content knowledge of parabola and exponential functions. The importance of this inquiry is directly concerned with the poor results of learner performance in mathematics in South Africa as evidenced in Grade 12 moderators' report (Department of Basic Education, 2017; 2018). The report suggests a need for a nuanced examination of the enablers or inhibitors of teaching and learning of mathematics, especially, at secondary school level.

Despite extensive literature on in-service mathematics teacher knowledge (Mji & Makgatho, 20016; Ball, Thames & Phelps, 2008; Bansilal, Brijlall & Mkhwanazi, 2014; Pornara, Hodgen, Adler & Pillay; 2015, Livy, Vale & Herbert 2016; Aksu & Umit, 2016; Pournara, 2016), there is limited research pertaining to pre-service teachers' common content knowledge of school mathematics concepts. Fewer, studies that have explored pre-service secondary mathematics teachers (PSMTs) subject matter knowledge of school mathematics topics (e.g. Ndlovu, Amina & Samuel, 2017) have argued for the development of specialised content knowledge. However, we are of the view that the development of specialised content knowledge is based on the individual schema development of common content knowledge of a particular mathematical concept. Before addressing the PSMT specialised content knowledge it is necessary to explore their common content knowledge. With that in mind, we quest to answer the following research question: *What is the nature of pre-service mathematics teachers' common content knowledge of parabola and exponential functions?*

Research into teacher Mathematical Knowledge for Teaching

South African studies exploring mathematical knowledge for teaching (MKfT) and like those in the international arena, have tended to focus on the common content knowledge of in-service mathematics teachers. The study by McAuliffe (2013) differs as it explored the development of primary school pre-service South African teachers' subject matter knowledge (SMK) of teaching early algebra. The findings revealed that the SMK for teaching early algebra was not fully developed as there was evidence of shortcomings in their ability to respond to three significant aspects: first, descriptions of procedures used by learners, second, to interpret learner productions and, third, to use learner errors to improve teaching. Pournara (2016) argued that working on school level mathematics can provide opportunities for pre-service secondary mathematics teachers to deepen their mathematical knowledge. He further, posit that the focus should be more on how one uses mathematics since it is assumed that s/he has the required mathematical knowledge. Lucus (2006) investigated both pre-service teachers and in-service Canadian teachers' conceptualization of composite functions. The findings showed poor conceptualization of the topic from in-service and pre-service teachers, thus suggesting lack of common content knowledge. Ndlovu et al. (2017) findings showed that pre-service teachers failed to interpret learners' errors on the concept of functions and inequalities.

The aforementioned studies revealed that there was lack of development in the PSMT's subject matter knowledge. Based on the assumption that PSMT have passed school mathematics, the aforementioned studies paid more attention on the issues related to the aspects of teaching school mathematics instead of common content knowledge. While we are not advocating for the teaching of school mathematics at higher level, we argue that in order to design alternative strategies to enhance PSMT subject matter knowledge, there is a need to understand the schema development of each knowledge strands needed in the development of subject matter knowledge. In this paper, we paid particular attention to PSMT common content knowledge. Adler (2017) argued that among other aspects of mathematics for teaching, attention to mathematical content is important.

Common Content knowledge

Ball, Thames and Phelps (2008) refer to common content knowledge as the knowledge teachers need in order to be able to do the work they are assigning to their learners. To provide a summary of what constitutes common content knowledge we listed the following. This is not claimed to be an exhaustive list.

- Calculate an answer correctly
- Solve mathematical problems
- Understand the mathematics you teach
- Recognise when students gives a wrong answer
- Recognise when a textbook is inaccurate or gives an inaccurate definition
- Use terms and notation correctly

Ball et al. (2008), posit that a teacher with developed common content knowledge should not make calculation errors, should pronounce mathematical terms accurately and be competent in solving problems.

To understand PSMT development of their common content knowledge, we explore their capabilities with respect to solving mathematical problems on parabola and exponential functions, procedural knowledge to accurately carry out the necessary procedures and use of notations. In order to explain the level of their performance we designed a genetic decomposition for parabola and exponential framework in line with APOS theory. The details of APOS theory can be found in Arnon, Cottril, Dubinsky, Oktac, Fuentes, Trigueros and Weller (2014). The authors posit that the design of the genetic decomposition to explore students' understanding of mathematical concepts can be informed by literature or the researchers' experience of teaching the topic. In this article, the designed genetic decomposition is informed by researchers' experiences of parabola and exponential functions.

Genetic decomposition of parabola and exponential functions

For his/her parabola function schema, the student with an action conception of the parabola would identify appropriate formulas, correctly read values from the graph/ equation, do step by step procedures, and correctly substitute values into the values or an equation. However, cannot think about the meaning of the answer, i.e. cannot see if the answer makes sense in relation to question asked and no step can be skipped.

At a process level, the student would interiorize the action into a process. This can be observed when the procedures are internalized. Actions are interiorized when the individual is able to make connections between the answer and the context of the problem, and make connection between concepts. At an object level, a student shows encapsulation of the process into an object when s/he can perform necessary actions and process to transform a function. Furthermore, perform actions and processes to transform the derivative to its original function

For his/her exponential schema, a student with an action conception, perform step by step calculations to determine the answer. At a process level, show the understanding of relationship between an exponential functions and its inverse. When these connections are internalized, the student can express the answer without explicitly writing down all the steps and use the knowledge of the functions to sketch its inverse. At an object level, the student is able to interpret data from the graph

METHODS OF DATA COLLECTION AND DATA ANALYSIS

The data reported in this study was collected in 2017 with second year PSMT. Data was collected by means of an activity sheet. Questions were taken from 2015 grade 12 final examination paper. The aim was to explore PSMT common content knowledge of functions and the choice of using questions from 2015 NCS paper was informed by the fact that the participants of this study wrote NCS in 2015. Since the participants qualify to major in mathematics in 2016, it suggests that their performance in the 2015 NCS was above 50%, which is the minimum requirement to major in mathematics in these three institutions. We therefore thought using the same question would share light as to how their common content knowledge of functions has evolved over the period of two years as they are at the institutions. Not all second year students consent to be part of the study, therefore the data presented here was collected from 179 students across three institutions who consent to be part of the study. In all three institutions coded as A, B and C almost the same number of PSMT consent to take part in the study. Thus fortunately making our analysis to be spread evenly. Data was analyzed based on our summary of what constitutes common content knowledge and the genetic decomposition. We first categorized the responses using the category derived from Ndlovu et al (2017) paper and we used the summary of what entails common content knowledge and genetic decomposition to generate themes for the discussion of findings.

FINDINGS

The items tested are presented below followed by the table showing how PSMT performed in each questions.

Item 1:The sketch below shows the graphs of $\boldsymbol{g(x) = x^2 - 3x - 10}$ *and* $\boldsymbol{h(x) = ax + q}$. *The graphs intersect at* ***B and D***. *The graph of g intersects the* ***x – axis at A and B and has a turning point at C***. *The graph of h intersects the* ***y – axis at D and the x – axis at B***.

Table 1: Questions used to explore PSMT common content knowledge

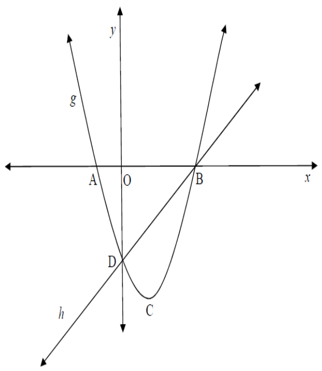
 	**1.1** Write down the coordinates of D. **1.2** Determine the coordinates of A and B. **1.3** Write down the values of a and q. **1.4** Calculate the coordinates of C, the turning point of g. **1.5** Write down the turning point of t, if $t(x) = g(-x) + 3$. **1.6** For which values of x will $g'(x).h'(x) \geq 0$?
Item 2: Given p(x) = 3^x .	**2.1** Write down the equation of p^{-1} , the inverse of p, in the form $y = \ldots \ldots\ldots$ **2.2** Sketch the graphs of p and p^{-1}on the same system of axes. Show clearly all the intercepts with the axes and at least one other point on each graph. **2.3** Determine the values of x for which $p^{-1}(x) \leq 3$

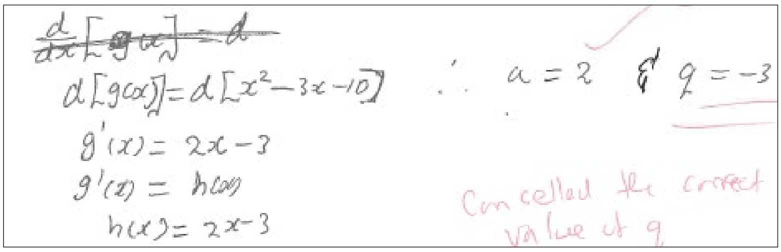
Table 2 below presents the analysis of pre-service teachers' Common Content Knowledge of parabola and exponential functions.

Table 2: Categorization of PSMT responses to parabola and exponential functions

Categorizing responses	Concepts tested								
	Questions on parabola						Question on exponential		
	1.1	1.2	1.3	1.4	1.5	1.6	2.1	2.2	2.3
Complete	157	168	139	114	37	45	11 8	53	7
Partial	2	4	24	35	56	17	1	58	39
Incorrect	17	3	10	19	60	66	34	27	65
No response	3	4	6	11	36	51	26	41	58

Point D is the point of intersection of the graph of g and *h* and *y* co - ordinate is the *y*-intercept of the two graphs. Identifying positioning of point D in the graph was critical in answering the question. To answer the questions PSMT needed to extract information from the equation of g since it was given. Reading information from the graph or the equation reveal the action conception of the concepts. Based on table 1, 87% of PSMT provided the correct response suggesting the development of action conceptions. Although the majority have the action conceptions, however, we were more interested in those who seem to lack the necessary conceptions since we expected all PSMT to provide correct responses. Since we did not conduct interviews, we cannot comment regarding no response category. What we notice as the source of error in this item was incorrect use of the notation. Instead of stating, the coordinate of D, PSMT wrote the y-intercept. All PSMT identify that D is the y-intercept but the item asked for the coordinate of D not only the y-coordinate. Similar knowledge required to answer item 1.1 was required for item 1.2 and similar errors were evident. In this case, PSMT wrote down the *x* coordinate only. Such error could be perpetuated by the fact that PSMT identify A and B as the *x-intercepts*, which was correct but the understanding that these values represent a point in the graph and the point is made of *x* and *y* seem to be not cognitively constructed. It could also be that the notation of expressing coordinates is constructed to be the same as stating either the *y*-intercept or the *x*-intercept.

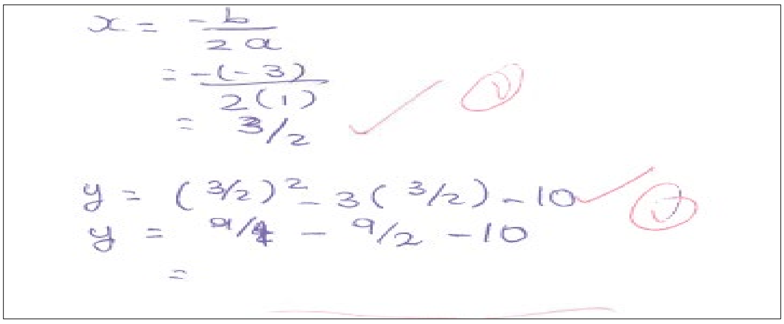
Seventy eight percent provided correct response to item 1.3. In this item "a" represented the gradient and "q" represented the y-intercept. All participants identified that the graph of *h(x)* was a straight line and therefore "a" represent the gradient and "q" represent the y-intercept. However, failure to accurately solve the answer and making sense of the answer was the main source of error. Secondly, while making connections between concepts in mathematics is encouraged because of lack of mathematical fluency some PSMT use complicated method to solve a simpler problem as shown in the extract below


$\cancel{\frac{d}{dx}[g(x)] = d}$
$d[g(x)] = d[x^2 - 3x - 10]$ ∴ $a = 2$ & $q = -3$
$g'(x) = 2x - 3$
$g'(x) = h(x)$
$h(x) = 2x - 3$
Cancelled the correct value of q

Extract 1 to item 1.3

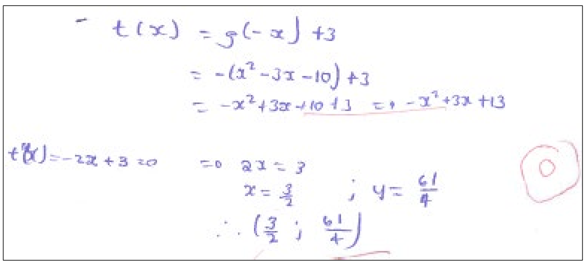
The student find the derivative of g(x) and conclude that g'(x) =h(x) which is not true.

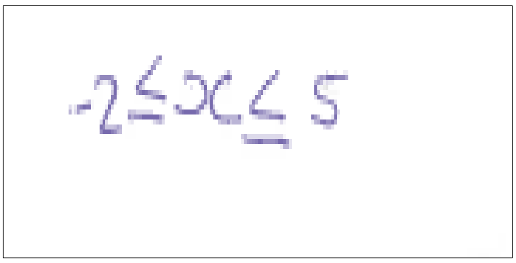
In item 1.4, 64% provided correct response. The main source of error in this item was procedural or lack of making sense of the solution.


$x = \frac{-b}{2a}$
$= \frac{-(-3)}{2(1)}$
$= 3/2$
$y = (3/2)^2 - 3(3/2) - 10$
$y = 9/4 - 9/2 - 10$
$=$

Extract 2 to item 1.4

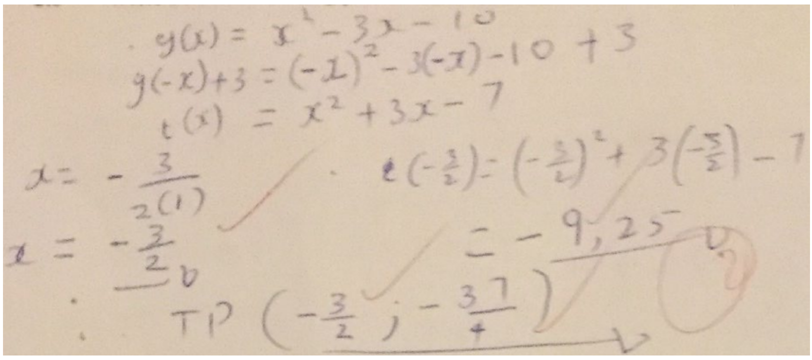
The student find the axis of symmetry, however failure to compute fractions hindered the student progress to determine the y-coordinate. Other students wrote C (0;10), using the y-intercept of the graph of g as the turning point. In item 1.5 and 1.6 we noticed a huge drop in the correct responses. Item 1.5 required the knowledge of transformation and 1.6 required making connection between the function and its derivative. In these two items, incorrect use of notation, failure to make connection between the function and the derivative, applying incorrect transformation, inability to make sense of their responses with respect to the context of the problem just do number grabbing and substitute was dominant. were the main sources of errors, thus suggesting gaps in PSMT common content knowledge of function concepts..


$t(x) = g(-x) + 3$
$= -(x^2 - 3x - 10) + 3$
$= -x^2 + 3x + 10 + 3 = -x^2 + 3x + 13$
$t'(x) = -2x + 3 = 0 \Rightarrow 2x = 3$
$x = \frac{3}{2}$; $y = \frac{61}{4}$
∴ $(\frac{3}{2} ; \frac{61}{4})$


$-2 \leq x \leq 5$

Extract 3 and 4 to item 1.5 and 1.6

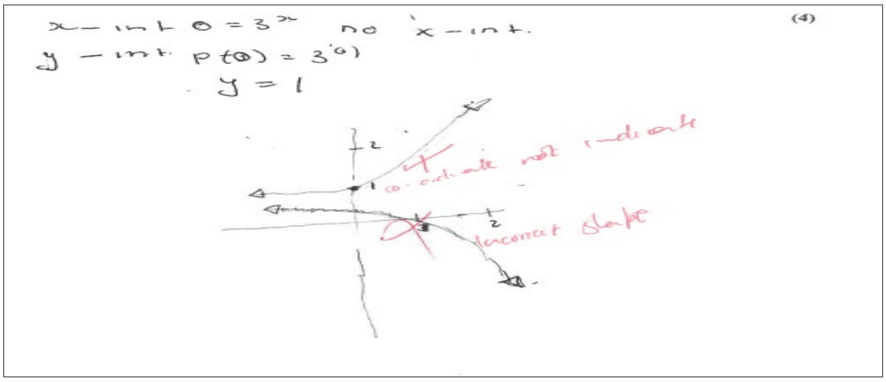
As in extract 3, PSMT substituted g(x) instead of applying the necessary transformation in the coordinates of the turning point. Even those who got it correctly could not apply transformation in the coordinates of turning point but needed to redo the whole procedure thus suggesting action conceptions



Extract 5 to item 1.5

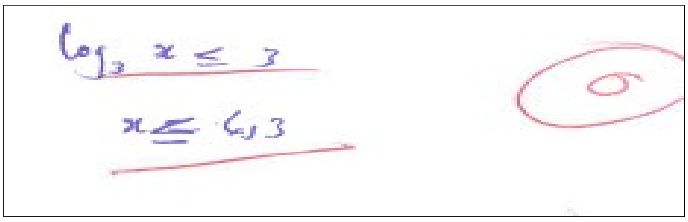
Number grabbing was also evident in item 1.6 as the students' use any numbers like the above response using the *x* intercept without trying to make sense of their response in relation to the question asked. The above findings supports the finding by Ndlovu and Brijlall(2015) who contends that pre-service teachers whose knowledge of concepts have not progress beyond action conceptions in many cases do not engage with what they write. Since the questions were taken from grade 12, same questions they wrote two years back, it was surprising to observe that we have a large number of PSMT who have not progressed beyond the action conception in many concepts needed in the schema development of a parabola.

In item 2, making connection between the function and its inverse was critical in answering all three-sub items. It was quite evident that the majority have not made those connections. Similar observation as in item 1 were evident in item 2, with a majority failing to answer questions correctly. The use of incorrect notation was dominant as the source of error. Moreover, in ability to make meaning of their responses with respect to the question asked was also dominant.



Extract 6 to item 2.2

The student did find the inverse algebraically in item 2.1 and the function was given, however, s/he could not sketch inverse graph. Even with sketching the function, the student first had to perform calculation and plot points suggesting that the process conception of the exponential graph has not been constructed. Similarly in item 2.3 PSMT provided any value of *x* without checking if it makes sense in the context of the problem and this was the question that was mostly not attempted by many PSMT

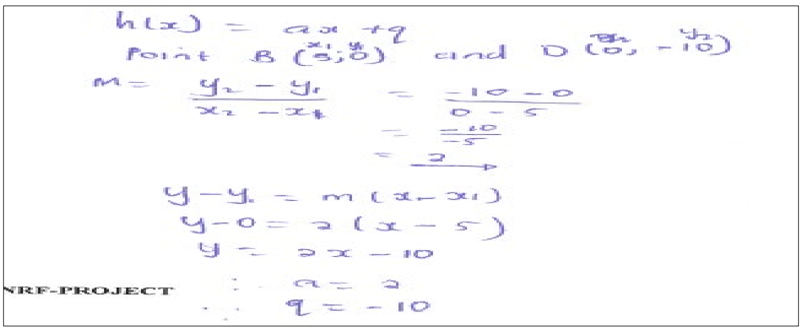


Extract 7 to item 2.3

These findings are consistent with Maharaj (2014), who contends that students do not check what they wrote. This is troubling in the context of PSMT as they are training to be teachers because it's one aspect they need to emphasize in the teaching. In none of the items where all participants provided incorrect responses thus suggesting lack of their common content knowledge and the majority shows action conceptions of the assessed concepts. Unlike Ndlovu et al (2017) findings that suggested PSMT only lack specialized content knowledge, these findings reveal that some PSMT do lack the common content knowledge as well.

DISCUSSION AND CONCLUSION

To answer our research question regarding the nature of pre-service mathematics teachers' common content knowledge of parabola and exponential functions, we further did an analysis to explore the number of PSMT who answered all the items correctly. Only 28 out of 179 PSMT we concluded that their common content knowledge of parabola and exponential function is intact because they provided correct responses to all the questions. However, not all of them we could say they have constructed the schema needed to conceptualize the concepts since regardless providing correct responses we concluded that they were operating at an action level as shown in the extract below:



Extract 8 to item 1.3

As it can be noticed the above response it's correct, however, this students failed to read the value of q from the graph. S/he performed all the steps in order to determine the value of "a" and "q" suggesting action conception as mentioned in the genetic decomposition. Based on the findings we concluded that with the majority of these PSMT as they progress with their tertiary education their common content knowledge of school mathematics is weakening. What transpired from their responses suggest that given a chance to teach they would struggle to solve some of the questions they would expect the learners to solve. As other scholars (Ball et al, 2008; Ndlovu et al, 2017), we suggest that teacher training institution need to direct attention towards development of PSMT subject matter knowledge of school mathematics if we aim to produce effective mathematics teachers.

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STUDENTS MISCONCEPTIONS IN ALGEBRA: A CASE OF NATIONAL CERTIFICATE (VOCATIONAL) LEVEL 2 ENGINEERING MATHEMATICS STUDENTS

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This paper reports on the misconceptions held by National Certificate (Vocational) Level 2 mathematics students in Algebra. Twelve students were selected using purposive sampling from a Technical and Vocational Education and Training (TVET) College in Tshwane, South Africa. The final examination scripts for these students were analyzed for misconceptions that were evident from their written responses to an Algebra question that weighs 25% of the paper. Donaldson's three categories of generic errors were used as a lens to interpret the students' misconceptions. The results reveal that structural errors are the most dominant amongst the three categories of misconceptions. These errors result from the student's failure to know the principles necessary to solve a problem, for example, conceptual errors.

INTRODUCTION AND BACKGROUND

Generally, many topics in mathematics require the application of algebra. It is therefore crucial for students to be proficient in Algebra in order to cope with these topics, including those that require deep mathematical thinking. This study is based on misconceptions that National Certificate (Vocational) (NC(V)) Level 2 engineering mathematics students have in Algebra. NC(V) programme is offered at TVET colleges in South Africa. NC(V) equips students with an extensive variety of knowledge and practical skills within specific industry fields. The *study fields* offered at the selected campus are Engineering and Related Design (ERD); Electrical Infrastructure and Construction (EIC); and Information Technology (IT). NC(V) was implemented for the first time in 2007 as an alternative to matric in a stream that prepares students for vocational curriculum as opposed to the academic curriculum (Grade 10 to Grade 12). Even though NC(V) has more than ten years since its inception, there is still a limited number of students who have completed the programme (Level 4) successfully so far. This is evident from the pass rates of the NC(V) Level 2 mathematics students of two TVET Colleges in Tshwane between 2011 and 2015 obtained from the Department of Higher Education and Training (DHET) when this study was conducted (refer to Table 1) at College A, which is the largest college in terms of enrolments. The pass rates between 2011 and 2015 are fluctuating around 30%. The data from the previous three years (2016 to 2018), however shows that the results are improving at both colleges.

Table 1: NC(V) Level 2 mathematics pass rates between 2011 and 2018

NAME	Percentage pass: 2011-2018							
	2011	2012	2013	2014	2015	2016	2017	2018
College A	40.5%	31.8%	28.5%	34.4%	32.3%	49.1%	68.0%	57.9%
College B	44.6%	19.9%	19.1%	43.8%	35.9%	53.5%	67.4%	50.8%
Average	**42.6%**	**25.9%**	**23.8%**	**39.1%**	**34.1%**	**51.3%**	**67.7%**	**54.4%**

Both researchers taught mathematics at TVET colleges at NC(V) and NATED[5] programmes. Researcher 1 (who got involved in NCV research from 2016) taught NC(V) for two years and NATED for two and half years. Researcher 2 taught NATED for five years; got involved in NATED research from 2004 to 2012 and got involved in Professional development for NC(V) (Level 2 to Level 4) and NATED (N4 to N6) lecturers in different provinces from 2013 to date. These researchers noted that NC(V) students have misconceptions and struggle with mathematics. The students might be performing poorly in Mathematics because of these misconceptions. The misconceptions, according to Luneta and Makonye (2010) weaken students' basic Algebraic skills like solving equations and factorisation, and ultimately compromise the performance of students. These reasons motivated researchers to investigate which misconceptions these students have in mathematics. In this paper, misconceptions that NC(V) Level 2 mathematics students have in Algebra are investigated. The research question is: *Which misconceptions do NC(V) Level 2 mathematics students have in Algebra?* Drawing from our experience at different TVET colleges, the problem of poor performance in mathematics at colleges has not yet been resolved. Given the fact that 30% is a pass mark, the low percentages are not good enough to allow students access to do topics at Level 3 and Level 4 (Calculus; Euclidean geometry and Linear Programming for example) that require deep mathematical thinking.

LITERATURE REVIEW

The results of a study on the acquisition of higher order skills in primary and secondary mathematics were that students discontinue studying higher level mathematics because they have difficulties with algebra (Pegg, 2010). Some of these difficulties are evident from the misconceptions and errors that students make generally. Misconceptions can be explained as fundamentally wrong views and values. These opinions can lead a student into committing a number of errors (Makonye, 2012). According to Luneta (2015, p.4) "misconceptions manifest in students' work as errors, which implies that errors are symptoms of misconceptions students possess". According to Luneta and Makonye (2010, p.158), "an error is a mistake, slip, blunder or inaccuracy and a deviation from accuracy". Misconceptions in this study refer to all wrong answers attributable to lack of knowledge or wrong belief by Level 2 mathematics students.

[5] Old curriculum which was offered before NCV programme was introduced. NATED curriculum is still existing at the TVET colleges as N1 to N6 programmes.

The results of the study by Mofolo-Mbokane (2012) revealed that, when interpreting graphs, students may choose the correct formula, but fail to apply it because of their misconceptions. The scholar further discovered that "the nature of errors made may be as a result of the students' lack of the mathematics register and probably because their knowledge in mathematics rules is superficial" (Mofolo-Mbokane, 2012, p. 285). The study by Egodawatte (2011) reveals that when students solve problems, they modify slightly and apply what they learned previously, be it a rule or an algorithm in a new situation. Mulungye, O'Connor and Ndethiu (2016) gave an example of expanding $(a + b)^2$ where students expanded it as $a^2 + b^2$. According to Mulungye et al. (2016), this is the misconception of the power of brackets. These scholars emphasise that teachers should bear in mind the students' misconceptions when they prepare their lessons and where possible use them when they present their lessons. In their study on algebra, Makonye and Matuku (2016) also discovered that students struggle to correctly apply the distributive laws and they struggle to multiply correctly (binomial or trinomial by binomial or trinomial). Students also collect unlike terms and join them together (conjoin) for example: $3 + x = 3x$. These researchers realised that students struggle to understand concepts such as unlike terms because of their failure to understand some of the algebraic terms. They further reveal that students have a tendency to wrap up expressions by combining unlike terms. Some of the examples they gave include: $2x^2 + 3x = 5x^2$. These scholars further state that students conjoin unlike terms and tend to take the highest exponent in the process.

In their study at which they investigated "the effect of probing on students in respect of their errors" Makonye and Khanyile (2015) compared the difference in performance prior probing and post probing. They reported that probing students regarding their errors, helped the students in dispelling misconceptions which lead to committing errors such as finding answers using the wrong mathematical rules; careless mistakes; dropping the denominator; confusing factors; inability to recognize the common factor; failure to factorize a trinomial; and failure to identify the lowest common denominator. These researchers claim that for students to be able to correct their misconceptions and improve their performance, teachers should make students aware of their errors. Beukes (2015) discovered that Maths N2 students struggle a lot when it comes to factorization, especially factorization of difference between two squares.

Makonye & Matuku (2016) witnessed students ignoring the equal sign where they changed the equation into an expression because they were confused by the statement: solve with the aid of factorization. Egodawatte (2011) discovered that students violate the property of equality where they put an equal sign in statements that were not equal. Bohlmann, Prince and Deacon (2017) also reported that students struggle with the differentiation between an equation and an expression. They claim that students fail to correctly apply matching procedures to the left and the right hand sides of an equation in order to solve the equation. The students lack in fluency when they deal with integers; fractions and algebraic conventions (O'Connor & Norton, 2016). These scholars also discovered that students could not obtain the final solution due to failure to understand the null factor, even though they show an understanding of factorisation.

Most of the literature discussed above involves studies that were conducted at secondary schools. Only two studies were conducted at TVET colleges. This reason makes this study of significance as it aims to address this gap. This study will equip lecturers with knowledge concerning the misconceptions that students have in algebra. The lecturers may use these misconceptions during their lesson preparations and presentations in order to help their students develop deep mathematical thinking.

THEORETICAL FRAMEWORK

It is not appropriate to think that the misconceptions that students have, emanate from what they were taught by their lecturers, students conceive these misconceptions by themselves (Confrey & Kazak, 2006). Confrey and Kazak (2006, p.201) point out that, "misconceptions are the strongest pieces of evidence for the constructive nature of knowledge acquisition, because it is highly unlikely that students have acquired them by being taught". They, however, agree that if the lecturer has a misconception, it is possible for him/her to pass it to his students. This study assumes the constructivist viewpoint (Confrey & Kazak, 2006) since it gives a clear and good description of how students comprehend mathematical knowledge including misconceptions.

Misconceptions in this study were categorized using Donaldson's (1963) error classification. Donaldson (1963), categorizes the errors that occur while students are learning mathematics as *executive errors, structural errors and arbitrary errors*. *Executive errors* are errors that occur when a student fails to carry out procedures or manipulations despite the student's understanding of the concepts required. An example of executive errors is failure to factorize (even though the student shows signs of knowledge of factorization). *Structural errors* are errors which may emanate from the student failing to know the principles necessary to solve a problem or lack of appreciation of the relationships involved in the problem, for example: conceptual errors. With *arbitrary errors*, a student may ignore some of the given information while s/he acts on the rest. Arbitrary errors occur when a student wants questions to only fit what s/he is acquainted with or what s/he knows, for example: their pre-knowledge.

METHOD

This is a qualitative case study where scripts of twelve students (from the largest campus from College A) were sampled from 150 students after writing their mathematics Level 2 final examination in 2016. Their scripts were analyzed for misconceptions Algebra (question 1), which was 25 marks of the 100 marks (25% of the paper). Algebra was chosen because students performed poorly in Paper 1. In particular, the performance was poor for Algebra and Functions (which require knowledge of algebra). It is also important to note that almost all topics in mathematics require application of Algebra. The researchers found it necessary to investigate misconceptions in Algebra. In the larger study (not the main focus in this paper) the same students were interviewed based on their misconceptions. The researchers ensured that the students from all study fields, also with different capabilities were included in the study (best, average and poor performing students).

DATA PRESENTATION AND RESEARCHERS' COMMENTS

Students were coded as: S_1 for Student 1 and so on until S_{12} for Student 12. In Table 2, the name of the category of misconceptions (*addition of unlike terms; failure to factorize a trinomial or difference of two squares; students' difficulties in solving simultaneous equations; failure to correctly apply the distributive law; equating expressions to zero; and negative sign ignored in the second term*) and the description used is given.

Table 2: Categories of misconceptions

	Category Name	Description
Misconceptions	Addition of unlike terms	Unlike terms are collected and added together.
	Failure to factorize a trinomial or difference of two squares	In this category, students showed their difficulties as far as factorizing a trinomial and a difference of two squares.
	Students' difficulties in solving simultaneous equations	This category illustrates how students struggle when they are supposed to solve simultaneous equations.
	Failure to correctly apply the distributive law	In this category, students could not correctly multiply $y(2y+1)$ when they were supposed to solve the equation $y(2y+1)=15$.
	Equating expressions to zero	This is a case where the students change expressions into equations.
	Negative sign ignored in the second term	In this category, students could not realize that when they take out a common factor which is negative, the negative sign will affect all the terms from which the common factor is taken from.

Some of these categories were adapted from literature. In their study "Learner Errors and Misconceptions", Luneta and Makonye (2010, p.161) used the categories: "Ignorance of rule restrictions; Incomplete application of rules; and False concepts hypothesized to form new concepts". The categories in this study were also influenced by the theoretical framework. In the theoretical framework, Donaldson's error classification emphasizes that there are three types of generic errors which consist of *executive errors, structural errors and arbitrary errors*. Table 3 to Table 8 present analysis and discussions of some of the findings in terms of the six categories in Algebra (Question 1 of the final examination). These tables consist of the name of the category, the question that was asked, example(s) of misconceptions students had and the researchers' comments.

Table 3: Addition of unlike terms

Students' Solution	Researchers' Comments
S_3: Factorize: $xy - 6ab + 2ay - 3bx$ 1.1.1 $xy - 6ab + 2ay - 3bx$ $= 2xay - 9abx$ $= -7xaybx$	The student added xy and $2ay$ to obtain $2xay$. The student continued to add $-6ab$ and $-3bx$ and got $-9abx$. This can be referred to as conjoining the variables because while adding unlike terms, the student joined the variables together. The conjoining of the variables continued in the last step where the student added $2xay - 9abx$ to obtain $-7xaybx$.
S_9: Solve for y: $y(2y + 1) = 15$ 1.3.1 $y(2y+1) = 15$ $y(3y) = 15$ $= 45y$	The student added unlike terms inside the brackets: $2y$ and 1 to get $3y$. The student then multiplied 15 which is on the right hand side of the equation with $3y$ inside the brackets on the left hand side to get 45 in the final step.

Table 4: Failure to factorize a trinomial or difference of two squares

Students' Solution	Researchers' Comments
S_1: Factorize: $x^2 + xy - 2y^2$ 1.1.2 $x^2 + xy - 2y^2$ $x^2 - 2y^2 + x = 0$ $(x+y)(x+y-x) = 0$ 1.1.3 $2x^2 - 5x - 12 = 0$	The first step suggests that the student is not familiar with this form of trinomials. The same student correctly factorized 1.1.3 This as a failure from the student to recognize the trinomial.
S_{11}: Factorize: $x^2 + xy - 2y^2$ 1.1.2 $x^2 + xy - 2y^2$ $\cancel{xy - 2y^2}$ $x(x + xy) - 2y^2$ $(x - 2y^2)(x + xy)$	In the first step, x is taken out incorrectly as a common factor. This student shows lack of knowledge of factorising a trinomial. The student factorized as if there are grouped terms. The last step involves taking out $(x + xy)$ as if it is common in both terms. This student appears to know factorisation as involving grouping and taking out a common factor.
S_1: Factorize: $a^2x^2 - b^2$ 1.1.4 $a^2x^2 - b^2$ $(a + x + b)^2 = 0$ $(x+a)(x+b) = 0$	These are wrong concepts. The student is applying the rules of exponents wrongly and at a wrong place. This shows lack of knowledge of factorizing a difference of two squares.

Table 5: Students' difficulties in solving simultaneous equations

Students' Solutions	Researchers' Comments
S_5: Solve the following system of linear equations algebraically. $2x - 4y = -9 \; and - 3x + 5y = 16$ 1.5 $2x - 4y = -9$ and $-3x + 5y = 16$ $2x - 4y + 9 = -3x + 5y - 16$ $2x + 3x - 4y - 5y + 9 + 16 = 0$ $5x - 9y + 25 = 0$ Let $x = 0$ Let $y = 0$ $5(0) - 9y + 25 = 0$ $5x - 9(0) + 25 = 0$ $5x - 25 = 0$ $-9y + 25 = 0$ $\frac{5x}{5} = \frac{25}{5}$ $\frac{-9y}{-9} = \frac{-25}{-9}$ $\therefore x = 5$ $y = \frac{25}{9}$ $y = 2{,}778$	In this case, the student transposed the constants so that the two equations are equal to zero. S/he then equated the two equations on the basis that they are both equal to zero. This is a misconception where the student confused this situation with making either variable the subject of the formula and equating the two equations on the basis that they are both equal to the same variable.
S_4: Solve the following system of linear equations algebraically. $2x - 4y = -9 \; and - 3x + 5y = 16$ 1.5) $2x - 4y = -9$ $-3x + 5y = 16$ $2x - 4y = -9$ ..... (1) $\times$ 5 $-3x + 5y = 16$ ... (2) $\times$ 4 $10x - 20y = -45$ $-12x + 20y = 64$ Subtract (2) from (1) $2x = 0 + \frac{19}{2}$ $x = \frac{19}{2}$ $10x - 20y = -45$ $10(\frac{19}{2} - 20y = -45$ $95 - 20y = -45$	The student in this case correctly multiplied equations (1) and (2) by 5 and 4 respectively. This was followed by addition of the two equations. The student claims to be subtracting (2) from (1) but the answer shows that the student added the two equations. This is a misconception. The student assumes that we always subtract one equation from the other when we want to eliminate one variable.

Table 6: Failure to correctly apply distributive laws when solving equations

Students' Solutions	Researchers' Comments
S_4: Solve for y: $y(2y+1)=15$ 1.3.1) $y + (2y+1) = 15$ $2y+1 = 15$ $2y = 15 - 1$ $2y = 14$ $\frac{2y}{2} = \frac{14}{2}$ $y = 7$	"y", which was outside the brackets has disappeared in the first step. It is difficult to believe that this is a slip, given the fact that the student did nothing to the equation in the first step. This is therefore categorized as false concepts.
S_{12}: Solve for y: $y(2y+1)=15$ 1.3).1) $y(2y+1) = 15$ $\Rightarrow 2y^2 + 1\cancel{y} = 15$ $\Rightarrow \frac{2y^2}{2} = \frac{14}{2}$ $y^2 = 7$ $y = \sqrt{7}$ $y = 2,656$	In the first step the student correctly multiplied y inside the brackets. The student later deleted y in the second term. The student might have realised that s/he has to deal with an unfamiliar equation.

Table 7: Equating expressions to zero

Students' Solutions	Researchers' Comments
S_1: Factorize: $xy-6ab+2ay-3bx$ 1.1.1. $xy - 6ab + 2ay - 3bx$ $(a+b)(2y-6+3x+xy)=0$ $(a+b)(2y-6+2x+y)=0$ $(a+b)(3y-2x-6)=0$ Factorize: $a^2x^2-b^2$ 1.1.4 $a^2x^2-b^2$ $(a+x+b)^2=0$ $(x+a)(x+b)=0$	The student included $=0$ in all the expressions that required factorization. This is an indication of the student confusing solving equations with factorization. The student also confirmed this misconception during interviews when s/he said: "*We always put equal to zero in mathematics. That is the reason I equated them to zero. Many problems in mathematics have equal to zero at the end*".

Table 8: Negative sign ignored in the second term

Students' Solutions	Researchers' Comments
S_{11}: Factorize: $xy - 6ab + 2ay - 3bx$ 1.1.1 $xy - 6ab + 2ay - 3bx$ $xy + 2ay - 6ab - 3bx$ $y(x + 2a) - b(6a - 3x)$ $(y - b)(x + 2a)(6a - 3x)$	In this case the student correctly grouped like terms. The common factor was correctly taken out in the first two terms. In the last pair, the student took out the negative sign in the first term only. 3 was also common in $(-6\mathrm{ab} - 3\mathrm{bx})$ but not considered.
S_5: Factorize: $xy - 6ab + 2ay - 3bx$ 1.1.1 $xy - 6ab + 2ay - 3bx$ $= xy - 3bx - 6ab + 2ay$ $= x(y + 3b) - 2a(3b + y)$ $= (y + 3b)(x - 2a)$	The student correctly took out a common factor in the first two grouped terms. In the second two grouped terms, the student took out the common factor of negative sign in the first term only. This is a misconception where the student does not see the negative sign as affecting both terms inside the brackets.

From the analysis, 70 errors were identified and categorized according to the misconceptions as discussed in Table 2. In responding to our main research question: *Which misconceptions do NC(V) Level 2 mathematics students have? It was found that* the most dominant category is *failure to factorize a trinomial or difference of squares* with 29 errors. This category is crucial as basic knowledge for many mathematics topics, including those that requires deep mathematical thinking. The second most dominant was *addition of unlike terms* with 15 errors. This kind of addition implies that students cannot differentiate like and unlike terms. The next two categories with 10 errors respectively are *students' difficulties in solving simultaneous equations* and *failure to correctly use distributive law*. The next category is *equating expressions to zero* with four errors. The last category is *negative sign ignored in the second term* with two errors. Most of the errors committed here show that students lack conceptual understanding.

DISCUSSION OF FINDINGS

The six categories established from our study are interpreted, related to previous studies and located within Donaldson error classification as *executive errors, structural errors and arbitrary errors*.

Addition of unlike terms

Addition of unlike terms was a challenge to the students. Students conjoin different variables when they see addition. They tend to eliminate the variable that is common when there is a subtraction sign. Students also tend to take the highest exponent if the variables are the same but have different exponents. Students perceive the operation signs between terms as an invitation to add or subtract terms. It appears as if students have a misconception that the final answer in mathematics must have one term. Makonye and Matuku (2016) also discovered similar results in their study where they reported that students do not understand the concepts of algebraic terms and find it difficult to understand concepts such as unlike terms. These researchers reported that students wrap up expressions where they combine unlike terms, example: $2x^2 + 3x = 5x^2$. These errors are categorised as *structural errors* according to Donaldson's error classification since the students show lack of knowledge.

Failure to factorize a trinomial or difference of two squares

Students find it difficult to factorize a trinomial. Out of twelve students, only four indicated some knowledge of factorizing a trinomial to a limited degree. Students have perception that factorization involves grouping only. These students would approach factorization of a trinomial by grouping the terms that contain a common factor first and take out a common factor in those terms. These students would then take what is in the brackets to be a common factor and disregard that the last term does not have what they contemplate to be a common factor. Similarly, Makonye and Khanyile (2015) discovered that students are unable to factorise a trinomial.

Students also find factorization of a difference of two squares to be a serious challenge. There was one question testing knowledge of a difference of two squares. Eight out of the twelve students got no mark. Students do not know what is expected out of them. Same as in this study, Beukes (2015) reported that students have misconception of factorization of a difference of two squares. In this category, students show that they have little knowledge of factorization even though they cannot correctly factorize. These errors are classified as *structural errors* according to Donaldson's error classification. The students have a misconception that factorization involves grouping which indicates a knowledge gap.

Students' difficulties in solving simultaneous equations

Students find it difficult to solve simultaneous equations. Two students achieved four full marks in this question and another two students obtained two marks. The rest of the students could not obtain a single mark. In general students have negative attitude towards simultaneous equations. They also have a misconception that elimination method only involves subtraction of equations. Students struggle a lot when they use elimination method. Those students who obtained full marks on this question used substitution method. Similar results as in this study were reported by Egodawatte (2011) who observed that students tend to misapprehend elimination method when they solve systems of equations. The researcher states that the students would try to avoid using the elimination method as much as they possibly can and resort to using the substitution method. Students showed that they have knowledge of solving simultaneous equations even though they struggled in the process. This category falls under *executive errors* according to Donaldson's error classification. The misconception that emerges in this category is that students assume that the elimination method only involves subtraction of equations.

Failure to correctly apply the distributive laws

Three students correctly applied distributive laws in the equation: $y(2y + 1) = 15$. Conversely, eight students correctly multiplied a binomial by a trinomial. This surprised the researchers since they believe it is more challenging to multiply binomial by trinomial compared to removing brackets in this equation. Perhaps the students are familiar with solving linear equations than quadratic equations. This can be seen from some of the responses from students where one student deleted y after multiplying out where y was correct. As observed from factorisation of a trinomial where students struggled, they might have tried to avoid factorizing a trinomial. In another case the, the "*y*" that is outside the brackets disappeared in the student's solution. This study concurs with Makonye and Matuku (2016) who discovered that students could not correctly apply the distributive laws where they struggled to remove the brackets. In this question, students could not factorize the quadratic equation after multiplying out. The results reveal that most of the students tend to ignore "y" outside the brackets as they factorize. This can be seen as *structural errors* according to Donaldson's error classification. In this category, it emerged that students have a misconception that equations are only linear. This is a sign of knowledge gap.

Equating expressions to zero

There is a student who equated four expressions that were supposed to be factorized to zero. Researchers view this as a misconception that originates from solving equations. This is an indication that the student cannot differentiate between expression and equations. The student confirmed this misconception during the interviews where s/he said "*all mathematics questions contain equal to zero"*. Conversely, Makonye and Matuku (2016) observed students disregarding the equal sign and converting the equation into an expression because they were confused by the statement: solve with the aid of factorization. This is also viewed as *arbitrary errors* since the student want the question to fit what s/he is acquainted with.

Negative sign ignored in the second term

There are two students who displayed a misconception where they took out a negative sign in one term and not in the second one. These students do not realize that a common factor should affect both terms and when the common factor is negative, both terms that the common factor is taken out of will be affected. Researchers advise that it is crucial to explain this using reversibility, where students will recognize that the factors will not multiply back. The researchers could not find anything in the literature related to students taking out a negative common factor in one term and not in the second term. In this case the students show lack of appreciation of the relationships involved when taking out a common factor. This error is therefore classified as *structural errors* according to Donaldson's error analysis.

CONCLUSION AND REFLECTIONS OF THE STUDY

The categories that emerged during data analysis include: *addition of unlike terms; failure to factorize a trinomial or difference of two squares; students' difficulties in solving simultaneous equations; failure to correctly apply the distributive laws when solving equations; equating expressions to zero; negative sign ignored in the second term.* Students mostly make *Structural errors* (addition of unlike terms; factorization involves grouping only; assumption that equations are only linear and negative sign not taken out in the second term when taking out a common factor). They also make *Executive errors* (they see elimination method to only involve subtraction) and *Arbitrary errors* (equating expressions to zero). These results indicate that structural errors dominate other categories of errors (4 out of 6). Structural errors are serious in the sense that they may impact on students' ability to do advanced mathematics topics in the TVET Level 3 and Level 4 curriculum which require deep mathematical thinking and conceptual understanding.

The results point to the difficulties that the sampled students have regarding Algebra. The researchers argue that these difficulties may generally hamper students to continue with mathematics topics that require deep mathematical thinking. It is imperative that TVET college lecturers deal with these types of misconceptions so as to enable their students the opportunity to learn. The complexity of the mathematics that these students are exposed to might be a reason why some students do not even make it to the next level. As it was pointed out by Pegg (2010), most of the students discontinue to study mathematics because they have difficulties with algebra. In this study, 422 students enrolled for NC(V) mathematics at Level 2. However, only 150 students wrote the final examination. This drop in numbers impact on the number of students that continue to do Level 3 and those that exist the system at Level 4. Another study must be done to find out why some of the students discontinued their studies. The intention of this study was not to generalize the findings, but to explore the nature on misconceptions found with this small group of students.

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AN EXPLORATION OF TEACHING STRATEGIES OF CORE AND EXTENDED CURRICULA IN TEACHING THE CONCEPT OF SOLIDS (MENSURATION): A CASE OF THREE SELECTED SCHOOLS AT MAFETENG DISTRICT IN LESOTHO

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The purpose of this study was to explore teaching strategies associated with core and extended curricula in the teaching of the concepts of solids (mensuration) in selected schools in Mafeteng district of Lesotho. A purposive selection of 30 learners (15 core and 15 extended) and three teachers in each of the three selected schools took part in the study. Qualitative case-study approach was used and interviews based on investigative task was conducted on the selected learners. The study revealed that teachers use direct traditional teaching approach as opposed to learners' centered teaching strategies. Learners showed lack of conceptual understanding. Therefore, it is recommended that teachers implement relational teaching to improve thinking and problem-solving skills.

INTRODUCTION

Lesotho as a commonwealth country has been implementing the education system offered by the University of Cambridge; which was the Cambridge Overseas School Certificate (COSC). Ansell (2002) argues that while curriculum reforms in Lesotho are intended to address the limitations of the colonial education, most reforms in their curriculum structure still mimic the key aspects of colonial education. The need to localize the 'O' Level curriculum and examinations has been a long-standing issue in Lesotho, since the early 1960s (Ministry of Education, Sports and Culture, 1982), when the weaknesses of the Joint Matriculation examinations, which were administered in South Africa, were noted. This led to the decision in 1961 to adopt the Cambridge Overseas Schools Certificate (COSC), administered by Cambridge University in the United Kingdom. The issue re-emerged during the National Education Dialogue in 1978 when problems associated with the COSC curriculum in the context of an independent Lesotho were noted. In the 1990s, the issue of localization of the 'O' Level curriculum became the central focus of national conferences and seminars. The most important of these was the 1995 seminar in which, for the first time, the meaning of localization in the context of Lesotho was clearly articulated. The report emanating from this seminar defined localization as " … taking charge and control of all activities and responsibilities over curriculum development and assessment" (Ministry of Education, 1995:18). The localization has brought the Lesotho General Certificate of Secondary Education in an effort to address the students' performance. The COSC was not only deemed difficult to learners but also outdated in addressing the new educational challenges.

This syllabus has separated the learners according to their mathematical ability titled core and extended. LGCSE (2015) identified that curriculum is meant for candidates who do not intend to pursue mathematically related courses, while the extended curriculum is suitable for those candidates who have an inclination towards mathematically related careers, as well as more able candidates. The Lesotho syllabus document for mathematics has suggested strategies for teaching learners of extreme abilities (both the gifted and low achievers). This has prompted teachers to devise strategies for teaching both core and extended content effectively. However, there are problems on this route, as in the school's capacity, the selection of learners, the timing of learners' selection, human resources and many day-to-day aspects of teaching and learning, which are influenced by the culture, and practices of individual schools. Thus, the aim of this paper is to explore teaching strategies associated with both core and extended pupils. Lekhetho (2017) attested that in Lesotho, students generally perform poorly in the Cambridge Overseas School Certificate (COSC) examinations, as can be seen by the low pass rates, which stood at 55.4% in 2012 Few students (less than 22% in 2012) qualify for tertiary education, and a dismal performance in mathematics and science, resulting in only a small percentage that secures admission into science-based programmes. LGCSE came in as a rescue to alleviate the poor performance which has affected the Lesotho students for many years. The overall performance, as displayed in Table 1, remain poor and shows no sign of improvement.

TABLE 1: LGCSE RESULTS 2014 – 2017

Year	Paper options	Total candidates	Above grade C	% above grade C	At grade C	% at grade C	At grade D	% at grade D	Below grade D	% below grade D
2014		18 986	738	3.886	1614	8.501	2229	11.74	14 405	75.86
2015	Core	9355	0	0	676	7.226	1383	14.78	1383	77.97
	Extended	9837	825	8.385	1309	13.3	1268	12.29	6435	65.4
2016	Core	8649	0	0	754	8.717	1306	15.1	4794	55.41
	Extended	9427	622	6.596	1253	13.29	1084	11.49	6486	58.59
2017	Core	11 093	0	0	565	5.093	1352	12.18	9176	82.70
	Extended	7510	360	24.35	1044	13.9	777	10.34	5329	70.97

From Table 1 it can be seen that Lesotho General Certificate of Secondary Education (LGCSE) was first introduced in schools in 2014. In that year the introduction of core and extended curricula was not implemented. Therefore, learners wrote the same component. In that year, only administrative changes to indigenize the examinations were carried out. Even so, poor performance continued: 75.86% of the learners failed to attain grade D. It can be noted that since 2015 the numbers of learners following the core curriculum have risen, while those on the extended curriculum have declined. This is because both learners and teachers have had to get used to the situation. It is equally true that the extended content is tougher and core is easier to manage. It can be seen that since 2015 no core learner has attained a higher grade than C. Also the quality of learners remains poor – no more than 825 learners have managed a grade above a C symbol. Despite this the overall number of learners who have passed has risen, which was why the two levels of mathematics teaching were introduced. This poor performance could be associated with the teaching strategies of core and extended mathematics curricula.

RESEARCH QUESTIONS

What teaching strategies should be explored to teach pupils in the core and extended curricula in the concept of solids (mensuration)?

- How does the choice of teaching methods influence core and extended pupils' solution strategies in mensuration?

RELATED LITERATURE

EFFECTIVE INSTRUCTIONAL STRATEGIES

Teaching methods or strategies are a pivotal stance in which teaching and learning are enhanced. PISA (2010, p. 20) states: "Teaching strategies refer to a broad range of processes, from the organization of classrooms and resources to the moment-by-moment activities engaged in by teachers and students to facilitate learning". The new challenge is now in the hands of the classroom teacher who has to decide what routes to take to help improve the understanding of mathematical concepts. However, these rely on a few critical factors. A teacher has to bear in mind that when deciding on the choice of teaching method; the topic, the nature of the objective to be met, the teaching environment, the pupils' background (age, prior knowledge and ability) as well as the type of instructional material that can be used. Topics are different and should be treated in a manner, which will intrigue the learners and induce more hands-on activities. Good teaching means that a teacher can move from style to style, strategy to strategy and learner to learner to create those climates and implement those strategies considered most conducive to mastering different kinds of objectives. This study will focus largely on the use of modelling as a method in teaching solids in this new syllabus. Mathematics is vital in enhancing competency, autonomy and interpretation in the mental processes of learners.

In his study, Mateo (2011) concludes that teaching strategies are not correlated with mathematics achievement but states further that good teaching strategies result in a more positive attitude and lessen anxiety towards mathematics. In fact, the way the teacher presents an activity or concept strongly influences the way the learners react to it. An effective teacher uses a variety of techniques and strategies to develop productive discipline and to motivate learners. The study by House (2001) on the relationship between instructional activities and the mathematical achievement of adolescents in Japan found that pupils tended to achieve better at mathematics when their teachers explain rules and definitions, and where do they come from. Similarly, pupils performed better in mathematics tests when their teachers more frequently solved an example related to the new topic. Therefore, it is clear that a teacher can make or unmake a learner in mathematics. Poor pupil performance can be linked to teacher input in terms of a failure to impart necessary knowledge, skills, attitudes and mathematical values or connections to pupils in their early days of learning mathematics. This study was conducted to make teachers aware of their immense contribution to the learning of mathematics, especially in Lesotho with the introduction of the (LGCSE) syllabus. A further motivation lay in the separation of learners in terms of mathematical ability. Here the implication is that no child will be left behind, as both groups deserve attention. This calls for proper teaching methods that match learner's ability. Thus this study explore teaching strategies associated to core and extended mathematics curricula in Lesotho.

Erebus International (2007), which evaluated a range of indigenous programmes, recommended that teachers recognise the individuality of pupils, provide a rich language environment, contextualise learning activities, identify what individuals know and need to know, and develop positive relationships with pupils. Sullivan, Youdale and Jorgensen (2010) identify factors contributing to pupil learning gains, including when teachers articulate the goals of teaching to the pupils and use thoughtfully sequenced activities that built on pupils' language and experience. This again puts a teacher in a position of high responsibility to design lessons that will inform learners in a way that largely brings about mathematical gain measured in pupils' language and experiences, which enhances the pupils' future prospects. Mathematics is both a subject and a language and should be treated with great caution.

According to Grouws and Cebulla (2000, p. 74) mathematics instruction should focus on meaningful developing ideas and highlighting their significance. This includes how the idea or concept of skill is connected in multiple ways to other mathematical ideas and forms of representations in a logically consistent and sensible manner. In other words, the instructional methods should incorporate basic intuitive ways of dealing with the mathematical problem either in small working groups or in the class as a whole. The role of the teacher should be well defined and aimed at helping learners realise their ability and learn new concepts.

As Feden and Vogel (2003, p. 20) put it, "active learning instruction is characterised by problem solving, students' participation and inquiry oriented teaching and learning strategies". Yoder and Hochelar (2005, p. 91) add that "the active learning instruction focuses on supporting the students towards the development of cognitive, affective and psychomotor behaviours as a learner, decision maker and community participant, with success measured in terms of student outcome". As perceived by teachers, the new syllabus, with its extended content, is bulky in terms of the number of topics to be covered in the space of two years, so it calls for the teacher to try all possible approaches or methods that will always involve learners, increase their participation and help to contextualize mathematical concepts. According to Zemelman, Daniels and Hyde (1998), the goal of teaching mathematics is to help pupils develop mathematical power, which enables the pupil to feel that mathematics is personally useful and meaningful, and to feel confident that he/she is able to understand and use mathematics. This challenges the beliefs and normal practices of teachers and tries to use methods that do not exacerbate the learners' appreciation of mathematics, but rather ameliorate and perk up mathematical confidence.

METHODS OF TEACHING AND LEARNING IN CORE AND EXTENDED MATHEMATICS

Many learning theories advocate for pupils to take a bigger share in the learning, so the study looks to use teaching strategies such as modelling as a teaching method that enhances learning through learner involvement. In general, there are two prevalent approaches to mathematics instruction. In skills-based instruction, which is a more traditional approach to teaching mathematics, teachers focus exclusively on developing computational skills and quick recall of facts. In concepts-based instruction, teachers encourage students to solve a problem in a way that is meaningful to them and to explain how they solved the problem, resulting in an increased awareness that there is more than one way to solve most problems. Most researchers (e.g. Grouws, 2004) agree that both approaches are important – that teachers should strive for procedural fluency that is grounded in conceptual understanding. In fact, the notion of numerical fluency, or the ability to work flexibly with numbers and operations on those numbers (Texas Education Agency, 2006), lies at the heart of an effective algebra readiness programme. According to social learning theorist Albert Bandura, "learning would be exceedingly laborious, not to mention hazardous, if people had to rely solely on the effects of their own actions to inform them what to do". Cai, Mamona-Downs and Weber (2005) assert that educators fail to expose their mathematics learners to multiple problem-solving strategies because they are inadequately prepared to deal with open-ended problems, doubt their ability to explain concepts and have the perception that multiple strategies and heuristics will only serve to confuse learners.

Modelling is a cyclical process of creating and modifying models of empirical situations to understand them better and improve decisions. The role of modelling and teaching mathematical modelling in school mathematics has received increasing attention as generating authentic learning and revealing the ways of thinking that produced it (Sung , 2012). The researcher also aims at finding whether some teachers do employ mathematical modelling in their classroom teaching and which topics can best be imparted by modelling. This arises from the inability of learners to view objects in three dimensions.

The National Council of Teachers of Mathematics (2014) argues that a modelling approach to the teaching and learning of mathematics shifts the focus of learning activity from finding a solution to a particular problem to creating a system of relationships that is generalizable and reusable. Models are systems of elements, operations, relationships and rules that can be used to describe, explain or predict the behaviour of some other familiar system. The modelling process begins with the elicitation stage, which confronts pupils with the need to develop such a model (Doer, 1997).

Engaging in this kind of modelling building is not seen as finding a solution to a given problem, but rather as developing generalizations that a learner can use and reuse to find solutions (Branford *et al*, 1996; Lehrer and Schauble 2000). To this end, I argue that pupils need multiple experiences that provide them with opportunities to explore the mathematical constructs, to apply their system in new settings, and to extend their model in new ways. Each of these stages of the model development process includes multiple cycles of interpretations, descriptions, conjectures, explanations and justifications that are iteratively refined and reconstructed by the learner, ordinarily interacting with other learners. This view of pupils' conceptual development through modelling is shaped by earlier research that posits a nonlinear, cyclic approach to model building (Doer, 1997). Learners need to be exposed to situations that help them to reuse the learnt skills on many similar or applying scenarios, so learning methods should involve learners in building up the concepts as well as creating formulæ. Such activities should be built to help learners visualize the solids from the inside out, from simple to abstract levels.

The National Centre for Educational Achievement (NCEA, 2009) examined higher-performing schools in five US states (California, Florida, Massachusetts, Michigan, and Texas) and determined that in terms of instructional strategies, higher-performing middle and high schools use mathematical instructional strategies that include classroom activities which:

- Have a high level of pupil engagement
- Demand higher-order thinking
- Follow an inquiry-based model of instruction – including a combination of cooperative learning, direct instruction, labs or hands-on investigations, and manipulatives
- Connect to pupils' prior knowledge to make meaningful real-world applications
- Integrate literacy activities into the courses – including content-based reading strategies and academic vocabulary development.

The challenge, because the learners are separated by the syllabus in terms of their mathematical ability, means teachers have to devise options or techniques to get most of the learners' attention. Such techniques of selection are not suggested in the syllabus to help teachers carry out this exercise. The core and extended strands show that there are low-and high-achieving learners who both demand the appropriate time to grasp and comprehend solids concepts. Low-achieving pupils may chronically experience lower-order instructional emphasis because educators see these pupils as "stuck" in the early phases of the learning process. By contrast, higher-achieving pupils, having mastered the basic skills, may be viewed as prepared to handle more complex learning tasks. A specific example of that general claim is manifested in Peterson's (1988) criticism of the traditional elementary school mathematics curriculum. It should be noted that elementary schools are lower classes offering grades 1 to 6 basic mathematical content before they progress to junior high school. According to Peterson, the traditional mathematics curriculum usually emphasizes a learning sequence according to which higher-order objectives are more appropriate for later stages in the learning sequence (like advanced geometry and algebra). Since only a few pupils progress to studying advanced mathematics, most will fall away before they get the opportunity to encounter truly higher-order instructional objectives.

TEACHING 3D OBJECTS (SOLIDS)

Solids include a variety of content and variations that need continuous testing at regular intervals, and such models should be used to gather learnt concepts at any reasonable stage of learning. The model of learning should be to engage in learning and be able to reflect on what pupils are learning at regular intervals (Additional Mathematics Syllabus, secondary three to four 2013). Thus, a circle of learning should be completed involving full engagement of the learners. This Singapore model of learning states that the circle may begin with a real-world problem leading to a mathematical model which then requires a mathematical solution to obtain a real-world solution (Mathematics Syllabus, secondary three to four 2013). On getting questions on the topic of *solids*, learners should:

1. Begin to understand and make some relevant assumptions;
2. Select an appropriate method to solve;
3. Interpret solution; and
4. Reflect on and improve the solution, if necessary.

Models not only give learners a role to play in their own learning of mathematics, but go further into the development of mathematical reasoning and visual skills. Learners who have been exposed to models tend to have a 3D eye. This is vital for helping learners in developing solution strategies in questions involving the topic of *solids*. This is vital in bringing about learners who are able to break down a worded *solid* problem into formulae and figures that connect the concepts. Modelling helps visualize the nature of the object(s) involved in the task, even when there are multiple or compound *solids*.

RESEARCH DESIGN AND METHODOLOGY

This paper seeks to investigate the teaching strategies associated with mathematics core and extended curricula in the teaching of the concept of solids in the Mafeteng district. In fulfilling this attempt, the researcher used a sequential explanatory mixed methods approach as guided and influenced by the literature review which, as seen earlier, dealt with the learners' abilities on the content and teaching methods. Three schools in the Mafeteng district were selected, each school providing 10 learners (five core and five extended) to give a total number of 30 Grade 12 mathematics learners.

The investigative task was chosen as an instrument to test the solution strategies of learners in solids concepts. Questions were asked from low to advanced level according to Bloom's taxonomy, testing the operation, manipulation, use of the units and application on solids. An investigative task was prepared for learners to test their solution strategies on the 3D objects.

In the three schools selected, only three teachers were chosen for the interviews. This selection of participants made a sample to assist in establishing qualitative evidence of the influence of core and extended content on the normal teaching of mathematics. Out of the 30 Grade 12 learners, 15 were core and the others extended. Data collected from them was processed and analyzed separately, but will at later stage be used to correlate the learning and teaching patterns in the core and extended curricula.

In each school, one Grade 12 mathematics teacher was chosen to take part in the study simply through teaching Grade 12 mathematics. Therefore, every teacher who handles core and/or extended mathematics in Grade 12 in the participating schools underwent an interview focusing on either the teaching methods or learners' errors in their solution strategies.

FINDINGS AND DISCUSSION

Good teaching of mathematics, especially on solids, should involve learning in both manipulations of the 3D objects and making mathematical connections to the formula derivation and use. Without these skills it is difficult for learners to display the mathematical skills which come through the development and use of the 3D eye. Continued manipulation of the 3D figures improves the ability to handle the algebra that has to do with perimeter, area, volumes and application questions as may be required.

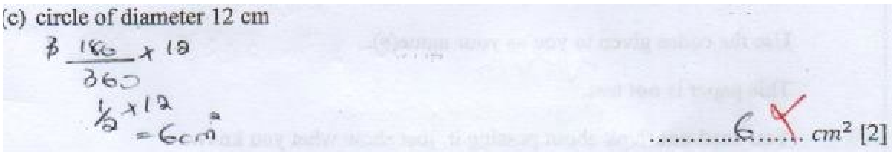
In this section, teachers were interviewed on the basic information and teaching methods in the teaching of the concept solids. These teachers were differentiated by the way they teach in terms of teaching a separate or a mixed class. Other schools have separated learners while others have kept them in one classroom. Separate classroom means core and extended pupils do not attended mathematics lessons simultaneously. Only the selected and relevant parts(responses) of the interview is reflected on and those will be on the issues pertaining to teaching experience, number of learners, teaching methods and the teaching of solids. These are chosen based on the research questions and the intention of the study. R will represent Researcher, T will represent teacher so that T1 will be Teacher No 1.

When asked about how do you normally conduct your lessons, when you teach, T1 said, " Normally no new approach. Just the normal way, showing learners what they do not know. When probed more by the researcher, T1 said, " Am saying there is no way the students were shown how these formulæ came about, to say this is the formula for the cylinder and this is how it was derived, to say one is for sphere, because it is never taught? It only comes with the paper and it is left for the students to use". This is evident when learners were asked to calculate an area of a circle of diameter 12 cm. Two learners responded as follows:

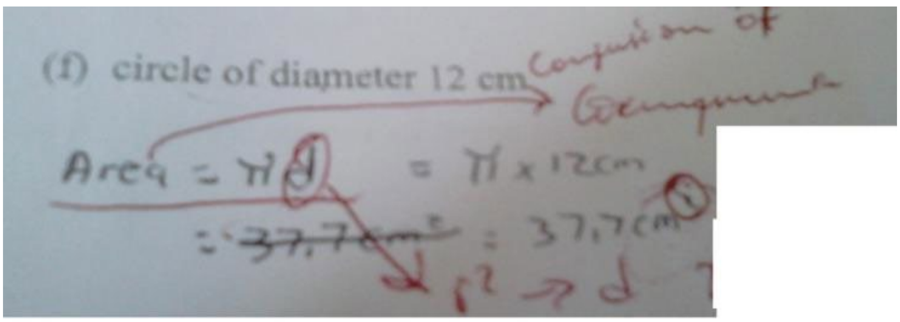
L1 from extended curricula

L2 from core curricula

L1 from extended class failed to deduce radius given the diameter. While L2 from core class brought in the notion of a semicircle and calculated this as if it was half of a circle. This is an error of misreading and in this case everything is correct apart from the $\frac{1}{2}$. L3 from the extended class shows that has no concept of the circle or when to use π. In answering Task 1, she wrote

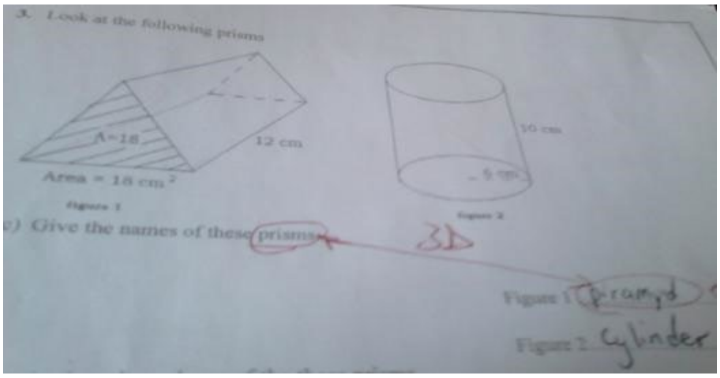


The figures 180 and 360 had no connection to the question and this clearly shows that this learner had a poor concept of the circle and took it as a semicircle, but still the fact that pi was not used raises a concern that the teaching process was not informing and conducive. This also suggest may have never taught how the formula of a circle is derived. When asked about the use of formula T2 said, "No we use the-e-e the ready derived formula". L4 from core class, as depicted below, used the formula for circumference to calculate the area of the circle, so the answer was wrong.



When asked why they do not derive formula, T2 indicated that "No it's time consuming" When asked how do he teaches the gifted learners, "*We may give them extra work on their own to meet …*"

It is obvious that teachers still go to class and carry out a routine without catering for learners' individual needs. This hinders the slow learners from being given the attention they fully deserve. Teachers continue to be traditional in their teaching by being central to both content and interaction (Machaba, 2018). As seen from T1, the formula are not treated as materials to learn but only to use. This is more than convincing confirmation that T1 uses traditional approaches. Despite this T3 is more engaged by having learner support to help bridge the gap. None of the teachers is prepared to derive the solids formula as they rely on giving the learners the formula for use without engaging with them about how the formula came about. In his study, Machaba,(2016) observed that learners lack the conceptual understanding of Area and Perimeter, they define a surface area of 2D shapes as " Multiply breath and length" without knowing where does it come from. In Task 2, learners were asked to give the names of the given prism. L2 from the core wrote:



The rest of the learners like L2 got it wrong and called it a pyramid, despite being given the lead in the question: "give the name of the prism". When teachers were asked specifically, how do they teach solids they said the following:

T1: We don't normally teach solids here, We are trying to be in a position to . . . finishing the syllabus. Maybe we have too many activities happening in our school so we don't have that enough teaching time to cover up or catch up with the syllabus topics.

T2: It's difficult because usually we use models . . . not models actually but pictures on their books which they are kind of two-dimensional. We sort of coming up with structures with three-dimensional solids that it's rare case that we ask them to build such models.

T3: Yaah, still basically myself on senior level, they do cuboids, basically prisms very well ehhhh, but when it gets to other components like the pyramids it's tough for learners.

The topic of solids is still not given the attention it needs as stipulated in the LGCSE mathematics syllabus document. In some schools, for example school 1, solids are not taught and this makes life difficult for learners when they are supposed to construe the mathematical connections in the solids. Even in schools where it is taught, learners are not given enough revelation on the breaking down and derivation of the formulæ (Brinker-Kent 2000). The syllabus has guidelines, but it can be seen that teachers do not follow such. Formulæ alone are not enough to create an attuned knowledge of a mathematical concept, as learners need proper teaching to help them develop the necessary skills in mathematics that will help see in 3D and make mathematical connections. It is also a general view that teachers are not well supervised, as they choose not to do certain topics. This is seen where school 1 has for some years now made it a tradition that solids are not taught. This practice deprive learners adequate skills to help improve thinking processes. In addition Turner (2010, p. 59), in his presentation at the Teaching Mathematics Conference, termed this "devising strategies", which he argued involves a set of critical control processes that guide an individual to recognize, formulate and solve problems effectively.

CONCLUSION

The study reveals that teachers – in their interviews – indicated the traditional ways of imparting knowledge to the learners. In the senior teaching of mathematics, none of the teachers mentioned modelling nor learner centred teaching strategies as a teaching method in mathematics. Furthermore, there was no mentioning of engaging learners into making nets of the *solids.* In addition, this did not help as learners failed to recognize the faces and names of the prisms.

From the analysis of the investigative task, 18 out of 30 correctly identified the name of the circular prism (cylinder) while only nine out of 30 identified a triangular prism. This was despite many of them calling it a "pyramid". On top of this, learners also failed to calculate the total surface area of the cuboid, basically confusing it with the cuboid's volume. This was much as expected, due to the fact that learners were not given the 3D eye into these figures; hence it was tedious for them to make sense of the shape. This failure not only showed a lack of understanding of the concept but also demonstrated that learners had no "connection" with the real world and what they were doing. Moreover, it is evident that learners' failed to link with the formulæ and was enough to prove that the teaching methods were not compatible with the learning processes so that learners could make deductions and connections with the real world.

Teachers need to move away from the traditional way of teaching, as it was observed that most relied on "book teaching" where topic 1 is followed by topic 2 – here teachers slavishly follow the order in the books without making connections on the nature and demands of the topic. Teachers should use more involving (interactive) methods to help learners grasp concepts well. Such methods should allow learners to take a central role in their mathematical learning. The teacher's role should be well-defined and unambiguous, so that the interventions are done at the appropriate time. In particular, the use of modelling in the teaching of solids (mensuration) should be implemented, as it has been observed that through modelling, learners are able to develop both solution-strategies and the 3D eye, which later result in good mathematical reasoning – which is so much needed at this level of learning. Good teaching methods are directly followed by good teaching tasks which ensure that proper learning takes place, which assesses what it has to assess without denying learners an exposure to new heights and learning variations.

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CONCEPTUALIZING THE LOGARITHMIC CONCEPTS: A CASE OF PRE-SERVICE MATHEMATICS TEACHERS

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An understanding of logarithms is pivotal in mathematics and often gives students serious difficulties. This study explored pre-service mathematics teachers' conceptualization of logarithmic concepts. Interviews were employed to study pre-service mathematics teachers specifically to ascertain how they link their previous mathematical knowledge to the understanding of logarithmic concepts. Constructivism was used as a framework for the analysis and the interpretation of how pre-service teachers conceptualize logarithms. The findings reveal that most pre-service mathematics teachers have limited understanding of logarithmic concepts because they cannot link their previous mathematical knowledge to understand the logarithmic concept. The findings in this study suggest a relook at the pedagogical strategies employed to foster pre-service mathematics teachers' understanding of these concepts.

INTRODUCTION

The nature of teaching and learning of mathematics has been a concern in mathematics education research (Klein, 2003). The better we understand how teaching and learning take place, the better we can plan the instructional methods and activities that will enhance the learning of mathematics as a subject. Several mathematics education research studies have looked at ways we can improve the teaching and learning of mathematics in school and at tertiary levels (e.g Berezovski, 2007; Biggs, 2011; Fiedler, Tröbst, & Harms, 2017). However, the learning of mathematics still remains a complex phenomenon (Öhman, 2015). Research into the teaching and learning of mathematics in general reveals that some students have difficulties in conceptualizing mathematical concepts (Naidoo & Naidoo, 2007).

There has been a growing interest in research in the learning and teaching of undergraduate mathematics over the past few years. Many scholars such as (e.g Nthontho, 2018; Pretorius, 2017; So, 2016) and others have conducted studies focusing on advanced mathematical thinking and pedagogies in the teaching and learning of mathematics. This body of research includes some work in the area of algebra, which is extensively pursued in research in recent years. However, the mathematics education literature regarding the pre-service mathematics teachers' knowledge of logarithms is lacking while vast literature exists on linear algebra. More specifically, little is known about how pre-service mathematics teachers conceptualize logarithms. Particularly, there is a paucity of research in this area.

PURPOSE OF THE STUDY

This study is part of a larger study on pre-service mathematics teachers' knowledge of logarithms. The primary aim of this research is to identify how pre-service mathematics teachers conceptualize logarithmic concepts by linking their previous mathematics knowledge to the concept of the logarithm. Constructivist theories are used as a framework to explore and understand if and how teacher link their previous mathematics knowledge to logarithmic concepts. The data collection method was interviews.

THE THEORETICAL BASIS FOR THE STUDY

Research in undergraduate mathematics education advocate for a unified theoretical perspective to understand how students construct their understandings of mathematical concepts (Krahenbuhl, 2016). Constructivism as a unifying theory seems to offer promise because it offers alternative pedagogical strategies for the classroom. This study is grounded in constructivist theory. According to this conception, learning occurs when learners make sense of new information by relating it to their prior knowledge (Ausubel, 1963). If the new knowledge does not fit well within the current network, the schemata may be restructured (Okanlawon, 2012). Once integrated, the schemata may be used to create new schemata in which old patterns may be linked to new.

Cognitive psychologists (Anderson, 2005; Sawyer, 2006) believes that learning is most likely to occur when an individual can associate new learning with previous knowledge. Learners work independently or in cooperation with others to internally generate unique knowledge structures. To solve a problem, students have to search for their knowledge structure from the knowledge that can be used to develop a solution pathway. An individual's knowledge is self-organized through various mental associations and structure.

Driscoll (2000) explains that constructivist theory asserts that knowledge can only exist within the human mind, and that it does not have to match any real-world reality. Learners will be constantly trying to derive their own personal mental model of the real world from their perceptions of that world. As they perceive each new experience, learners will continually update their own mental models to reflect the new information, and will, therefore, construct their own interpretation of reality. Accordingly, Bada and Olusegun (2015, p. 67) claim that "constructivists believe that knowledge is built up by students as they abstract from and reflect on the mental and physical actions of their experiences and their environment".

Constructivism's central idea is that human learning is constructed, that learners build new knowledge upon the foundation of previous learning (Bada & Olusegun, 2015). This view of learning sharply contrasts with one in which learning is the passive transmission of information from one individual to another. There are two important notions around the simple idea of constructed knowledge. The first is that learners construct new understandings using what they already know. There is no tabula rasa on which new knowledge is etched. Rather, learners come to learning situations with knowledge gained from previous experience, and that prior knowledge influences what new or modified knowledge they will construct from new learning experiences (Phillips, 1995).

Development of the constructivist view of learning has resulted in modifications of teaching design in many science classes (Duit, 2016). An innovative constructivist teaching program usually implies a modification of teaching tasks and strategies, learning tasks and strategies, and the criteria for learning achievements. Therefore, this study is framed using the constructivist theory with a special emphasis on connecting the previous knowledge in solving logarithmic problems.

REVIEW OF RELATED LITERATURE

K. Weber (2002) conducted a study to describe instruction proposed to facilitate student learning of the concepts of exponents and logarithms. In his work, he hypothesized that students first need a process understanding of exponentiation. In other words, they need to learn to understand exponentiation as real valued quantity. He argued that the most plausible way that a student can learn to understand real-valued functions is to understand first, exponential functions with their domain restricted to the natural numbers. Without this initial understanding, students are unable to apply the concepts in novel situations in a meaningful way.

Kastberg (2002) conducted a study to develop descriptions of students' understanding of the logarithmic function and ways of knowing how they use to investigate problems involving the logarithmic function. According to Kastberg, a student understands a concept when his or her beliefs are consistent with those held by the mathematical community. Kastberg's study also suggested that students believed that performance is understanding.

Kenney (2005) investigated how college students interpret logarithmic notation and how they use these understandings to solve problems that involve logarithms. In his study, Kenney explains that students who can utilize this framework think unambiguously about the dual role of the symbolism, while the less able, rely on memorized procedures evoked by the symbolism encountered.

Weber (2016) argues, with a broad basis in the literature, that logarithms have a reputation for being difficult and inaccessible. He points at the presentation of logarithms as the inverse of exponential functions as a probable cause of this difficulty. Perhaps another part of the reason for this difficulty is that the students do not perceive logarithm as relevant to them.

In another study by de Gracia (2016), most students liked to skip certain important steps when working with logarithms. Moreover, among the laws of the logarithm, the first and second laws emerged to rank second and third respectively, with the most frequent number of mistakes committed by the student-respondents. In their study of the logarithm, students mixed up exponential and logarithmic rules.

It is important to highlight that so far, research has not been carried out in the area of logarithms in both high schools or universities in South Africa. There appeared to be very limited sources in the literature on the subject. This study therefore, will answer the question, “how do pre-service mathematics teachers conceptualize logarithmic concepts”?

METHODOLOGICAL APPROACH

This study is grounded in the interpretive paradigm. The focus was on how the pre-service mathematics teachers conceptualize logarithm. Therefore, a qualitative approach was used to collect and analyze data. Again, in mathematics education research, some believe that excessive reliance on statistical measures strips away context and meaning (Orcher, 2016) in the research. Therefore, alternative approaches to mathematics education research employ a qualitative methodology that also brings rigour and effectiveness to the research process.

The interpretive paradigmatic approach will underpin the understanding that reality is multiple, contested and is negotiated in human experiences and perceptions (Lewis, 2015). It enabled the study to elicit thick and rich meanings of reality that the pre-service mathematics teachers gave to their experiences on how they conceptualize logarithm. Scotland (2012) asserts that interpretivists believe that there is no one particular right or exact approach to knowledge. This suggests that there is no specific answer, but answers are subject to people’s previous knowledge or experiences.

In this research study, a qualitative methodology that made use of interview as a method of data collection was adopted. The interview was used as a method of data collection because it allowed me to obtain detailed information about pre-service mathematics teachers perception when linking their previous knowledge to solving logarithmic problems through probing questions. The data was collected using purposive sampling from one of the universities in KwaZulu-Natal. Nine pre-service mathematics teachers who enrolled in one of the universities in KwaZulu-Natal volunteered to participate and interviewed for this study. In the interviews, they were asked a wide range of questions: Pre-service mathematics teachers were asked if there are previous knowledge that they know which can be applied in the understanding of logarithm. The pre-service mathematics teachers were also asked an open-ended question designed to probe how they fully understand logarithmic concepts using their previous knowledge and how they can use their previous knowledge to solve logarithm problems. The interview lasted for 45 minutes with each participant.

PRESERVICE MATHEMATICS TEACHERS’ CONCEPTUALIZATION OF LOGARITHMIC CONCEPTS

This section reports the result of the study in which we analyze pre-service mathematics teachers’ conceptualization of logarithmic concepts within the context of the constructivism theory.

The main finding of this study was that seven out of the nine interview pre-service mathematics teachers could not link their previous knowledge to logarithm when it was introduced to them. Furthermore, they could also not apply their previous knowledge to solve logarithmic problems. Pre-service mathematics teachers' responses to some of the questions are presented below.

Did you have any prior knowledge that links to logarithm? What is it?

This was an open-ended question designed to understand pre-service mathematics teachers' previous knowledge that linked to the logarithm. Three pre-service mathematics teachers said, "*the knowledge of exponent*". The rest of the pre-service mathematics teachers' responses were varied, and somewhat distinctive. Examples of some of these responses are given below:

Student1: *Am not sure.*

Researcher: *You are not sure of the prior knowledge or you are not sure of what I meant by prior knowledge?*

Student 2: *I think logarithm is linked to financial mathematics. I remember using logarithm in financial mathematics.*

Student 3: *No, there is no prior knowledge.*

Researcher: *Can you elaborate on that?*

Student 3: *I don't think so. I don't know.*

Besides the first three pre-service mathematics teachers, no other pre-service mathematics teacher demonstrated using prior knowledge before logarithm was introduced to them. If nothing else, this indicates that these pre-service mathematics teachers do not understand fully what logarithm is.

Looking at this question, $log_{12}(3-x)+log_{12}(2-x)=1$*, what other knowledge will you require to be able to solve the question correctly?*

Answering this question correctly required that the pre-service mathematics teachers would know the laws of logarithm and how to convert logarithmic equation to the corresponding exponential equation. It also requires the knowledge of quadratic equations, linear equations and an idea of inequality in the sense that logarithm of any number less than or equal to zero is undefined.

Only one pre-service mathematics teacher was able to explain all the previous knowledge required. She did so by solving the question herself. Her response was given below:

Student 5: *You see, to solve this question, you will first apply the multiplicative law of logarithm so that the left-hand side of the equation can become one term, then you change the logarithm to the exponent. After that, the equation will become quadratic when simplified. So, you will solve the equation to obtain the value of x. But you have to test to see which value of x is a solution since the log of zero and negative numbers are undefined.*

Researcher: *So, you need the knowledge of quadratic equation and inequalities too, not just logarithm?*

Student 5: *Yes.*

The eight other pre-service mathematics teachers said, "*the knowledge of logarithm laws*". Two of them included knowledge of exponents. This also shows that the majority of pre-service mathematics teachers interviewed cannot link their previous knowledge to the solution of a logarithm problem.

Look at the solution given below, what are the knowledge or skills applied in solving the question?

The solution for the equation $27^{\log_3 x} = 8$ is given below.

$$log\, 27^{log_3 x} = log\, 8.$$
$$log_3 x\, log\, 27 = log\, 8.$$
$$log_3 x = \frac{log\, 8}{log\, 27}.$$
$$x = 3^{\frac{log\, 8}{log\, 27}}.$$

All the 9 pre-service mathematics teachers answered that the knowledge of logarithm laws was applied in solving the question. Only two students were able to include the knowledge of conversion of the logarithm to the exponent and solving of linear equation. A representative response is given below

Student 2: *The knowledge of laws of logarithm*

Researcher: *Which of the laws are applied?*

Student 2: *Am not too sure.*

Researcher: *Is there any other previous knowledge that was applied which is not the laws of log?*

Student 2: *Emmmm, (silent), no.*

Researcher: *Having seen this example, can you be able to solve a related question like this?*

Student 2: *(Silent). Am not too sure. This sum looks complicated.*

Student 7: *The knowledge of laws of logarithm and exponent.*

Researcher: *Which of the laws are applied?*

Student 7: *Aaaaah (silent), eish, I don't remember the laws, but I guess it is related to the exponents. We didn't do much log problems.*

Researcher: *Is there any other previous knowledge that was applied which is not the laws of logarithm?*

Student 7: *Eish, (silent), looking at this solution, I can only see log everywhere.*

Researcher: *Having seen this example, can you be able to solve a related question like this?*

Student 7: *No. I wouldn't lie. To be sincere, I don't think I can solve log sum. I might be able to use log to solve financial maths. I like money (*laughs*)and I try to do my best when it comes to finance.*

Student 5: *The knowledge of logarithm and conversion of the log to the exponent. You equally need to know how to solve the linear equation so that you can proceed from step 2 to 3*

Researcher: *Which of the laws of logarithm was applied?*

Student 5: *I think power law was applied in step 1 to get to step 2.*

Researcher: *why was log introduced to both sides of the equation?*

Student 1: *Am not too sure.*

This clearly indicates that the majority of pre-service mathematics teachers' conceptualization of logarithmic concepts through the lens of constructivism, specifically linking to their previous knowledge, is limited.

SUGGESTIONS FOR PEDAGOGICAL PRACTICES

Before the introduction of any formal instruction on logarithms, instruction should focus on the need to link the idea of the previous knowledge. The formal definition is then not an isolated concept. Constructivist theories which focus on prior knowledge maintains that learning experiences occur as students actively construct or reconstruct new schemas. If the new knowledge does not fit well within the current network, the schemata may be restructured (Okanlawon, 2012).

Pedagogical strategies that focus on developing pre-service mathematics teachers' thinking are critical if the goal of instruction is for them to know what to do. With little more than instrumental understanding (Ansah, 2016), pre-service mathematics teachers may fail to see the link between the previously acquired knowledge with the logarithmic concepts. The curriculum developer will therefore be tasked to present in context the cognitive growth possible, leading ultimately to meaningful mathematical thinking in which previous knowledge plays an appropriate part.

CONCLUSION

This paper explored pre-service mathematics teachers' conceptualization of logarithm concepts. We also analyzed pre-service mathematics teachers' understanding of logarithm in the context of our theory only to find that majority of pre-service mathematics teachers (7 out of 9) cannot connect their previous knowledge in understanding of logarithmic concepts. Hopefully, employing our suggestions will improve the teaching and learning of logarithms.

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ANALYSIS OF STUDENTS' UNDERSTANDING OF RADIAN MEASUREMENTS

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This paper reports an exploration of students' understanding of radian measurements. The participants of the study were a group of at risk thirty students registered for mathematics among first year electrical engineering in a University of Technology. The study employed a qualitative illustrative case study approach and collected data through documents analysis that is students' written work. This research uses Insight Algebraic framework to analyze and interpret the data collected. The study explores the components of a circle such as a centre, circumference, diameter, tangent, major arc, minor arc, central angle, sector and segment. The understanding of the components of a circle were considered as an fundamental for developing understanding of radian measurements among first year electrical engineering students. In order for students to understand radian measurements mathematical concepts such as properties of a triangle, substitution, π, semi-circle and revolution are required to be developed as preliminary understanding of radian measurements. Findings of the study reveal that first year university students lack understanding of essential geometry concepts in the learning of radian measurements. The study recommends that developing baseline assessment will assist lecturers to develop learning activities that will facilitate smooth learning of radian measurements.

Keywords: radian measurements, conjectures, circle geometry, components of a circle

INTRODUCTION AND BACKGROUND

This study focuses on sense making in development of how formulae that are applied to solve problems in radian measurements were invented in order to build meaningful understanding among students registered for mathematics in their first year level in Electrical Engineering field. Several researchers claim that trigonometry is an example of a topic that lacks coherence in mathematics education (Akkoc, 2008; Thompson, Carlson & Silverson, 2007). My view is that, this is analogous of radian measurements in lack of coherence. Radian measurements are only introduced in first year level in a University of Technology and there is no clear link between the content taught in radian measurements and content studied in secondary schools in South Africa.

The poor performance of students in trigonometry is associated with fragmented angle measure understanding that inhibit their ability to construct flexible trigonometry functions (Moore, 2013). Moore (2013) asserts that "teachers' and students' angle measure understanding often lacks meaningful connections to arcs" (p.226). The study explores students' understanding for the components of a circle as pre-requisite for understanding radian measurements in the field of engineering.

This is similar to the South African curriculum, Curriculum and Assessment Policy Statement (CAPS) document (2011), it needs expansion of what should be taught in circle geometry such as area of a sector and area of a segment.

Learning a new topic in mathematics is a problem to many students in all levels of schooling. Students enter University of Technology with one understanding that angles are measured in degrees. The University curriculum in South Africa where the study was conducted, in preparation of introducing calculus proposes introduction of radian measurements before starting trigonometric ratios, functions and calculus. Most of the first year students who are registered for mathematics come across the word 'radian measure' for the first time in measuring angles. The introduction of radian measurements creates a confusion to many students as to why radian measurements instead of degrees in measuring angles. Max (2012) argues that the use of radians made it easier to measure angles that involved more than one rotation. This necessitates lecturers to demonstrate a link between angles and radians. Mathematically a radian is defined as $\theta = \frac{s}{r}$ rad, where the symbol $\equiv$ means "is defined as". In the formula, θ represents the angle being measured, s represents arc length, r represents radius, and rad means radians. Look at figure 1:

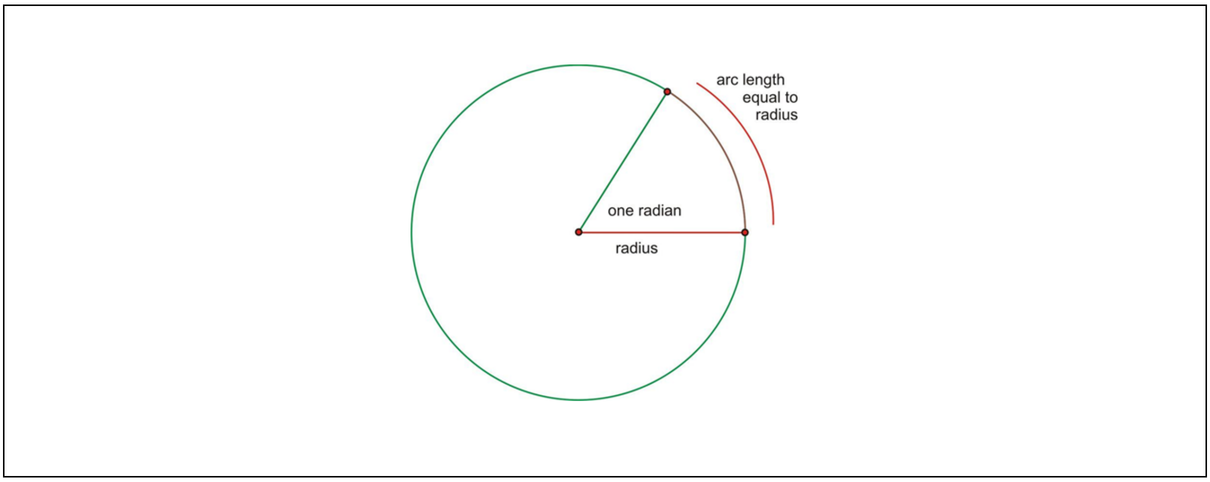

Figure 1: shows relationship of a ratio; arc length (s) = radius (r) when $\theta = 1$ rad

This study explores difficulties encountered by first year students in the learning of radian measurements. This encompasses to teach students how mathematicians develop formulae in the teaching of radian measurements. Students' understanding and enjoyment of mathematics depends on the development of mathematical reasoning (Makonye, 2014). Mathematical reasoning is a critical skill that enables a student to make use of all other mathematical skills.

In order for students to demonstrate comprehensive knowledge of radian measurements they are expected to have meaningful understanding of components of a circle. Students should be able to retrace formulae that are used to solve mathematical problems.

It should be stated at the introductory stage of radian measurements that components of a circle are fundamental in developing students' understanding of formulae used to find the total surface areas of sectors and segments in unit circles. This can assist students to gain an understanding of the relationship between radian measurements and the components of a circle.

Inclusion of circle-geometry in a secondary school South African mathematics curriculum is not consistent. As a result, there are students that find it difficult to apply understanding of formulae deduced from circle components. Despite the fact that a circle is introduced in the lower grades of schooling, detailed analysis of components of a circle are not dealt in South African curriculum. In lower grades, only recognition of a circle and the calculation of an area of a circle are taught. Students were only taught that area of a circle is equal to πr^2. Further components of a circle such as relationship of a diameter, radius, sectors, segments, arcs, chords, tangents are only taught in Grade11. All the same, calculations of areas of sectors and segments, are omitted in the mathematics curriculum recommended for secondary schools in South Africa.

Students who study mathematics at the entry level in a university struggle to grasp radian measurements and its applications. Students who demonstrated poor performance in mathematics in the field of engineering triggered lecturers to do baseline assessment to determine what the students know from secondary school level.

This study sought to answer the following questions:

1. What are difficulties encountered by first year students in the learning of radian measurements in an Electrical engineering field?
2. What are teaching strategies that should be employed by lecturers to address encountered difficulties?

LITERATURE REVIEW

It is reported that several students, teachers and educators have difficulty in reasoning about trigonometry, trigonometric functions and other related topics (Brown, 2005, Fi, 2003; Thompson, Carlson & Silverman, 2007; Weber, 2005). Moore (2009) asserts that "few studies have inquired into the reasoning abilities and understanding that are foundational in developing trigonometric understanding" (p.1).

This is analogous to the understanding of radian measurements with regard to the reasoning abilities and understanding that are foundational in developing radian measurements understanding. In order for students to develop understanding of trigonometry, trigonometric functions and radian measurements, students need to master triangle trigonometry and circle trigonometry. The understanding of circle trigonometry rely heavily on understanding of circle geometry.

The understanding of circle geometry entails comprehensive knowledge of its components, notably diameter, circumference, arc length, sectors segments, central angle, radii and chord. In order for students to understand trigonometry, trigonometric functions, and radian measurements mathematics teachers should develop lessons that connect these concepts to students' understanding of circle geometry.

The understanding in the learning of mathematics depends on how teachers plan their lessons. Lessons should be planned in such a way that they develop logic in students' mind. Moore (2009) asserts that "the difficulties that students encounter in developing coherent trigonometric understanding is likely multifaceted" (p.2). He further cites Thompson (2008) that "trigonometry may be a difficult topic for students due to an incoherence of previous understanding and a lack of developed reasoning abilities that are foundational to building understanding of trigonometry" (Moore, 2009, p.2). The current study deals with analysis of components of a circle as a pre-requisite for an understanding of radian measurements among first year students in Electrical Engineering in a University of Technology in South Africa.

In analysis of South African curriculum, Department of Education Curriculum and Assessment Policy Statement (CAPS) document (2011), differs with U.S. countries where both radian measurements and trigonometry are recommended to be taught in secondary schools. In a South African context, triangle trigonometry, circle trigonometry and circle geometry are recommended in secondary schools. However, radian measurements is only introduced at the first year University level in the field of engineering. As a result students who graduated matriculation in South Africa struggle to grasp understanding of radian measurements in a University.

There are few studies conducted on developing students' understanding of angle measure in degrees and measure of radian measurements. The study conducted by Moore (2009) reported that "two studies characterized by pre-service and in-service mathematics teachers in Turkey as holding understanding of radian measurements dominated by degree measure ; when given radian measurements, the teachers converted these measures into a number of degrees in order to attribute a meaning to the radian measurements" (p.2).

Akkoc (2008) reports that pre-service teachers claim that radian measurements are given in terms of π, leading teachers to interpret 30 as a number of degrees in expressions such as $\sin(30)$. Akkoc (2008) further suggests that impoverished radian measurements understanding likely contribute to teachers and students difficulties in trigonometry.

Comprehensive knowledge of radian measurements depends on the understanding of fundamental concepts such as circle and its components. Understanding of a circle and its components serves as a foundation to construction of the definition of radian measurements and establishment of clear distinction between radians and degrees. This is an essential tool to equip students with the necessary vocabulary required to comprehend radian measurements. Radian measurements require students to understand all components of a circle together with their symbols, namely, central angle (θ); circumference (C), arc length $(\frown)$, diameter (d), radius (r), and the Greek letter pi (π).

The letter π represents a number, a mathematical and physical constant which originated as the ratio of a circle's circumference to its diameter (Perkovac, 2016, p. 1899). Perkovac (2016) characterizes the number π as a transcendental number that is a fundamental mathematical constant with the decimal expansion $3{,}141592653...$ A central angle is the angle formed when two radii of the same circle meet at the centre. Most of the time, it is denoted by θ . Perkovac (2016) argues, "while π is a number, pi is a quantity defined as the ratio of the circumference of a circle (2π) to its diameter $(d = 2r)$" (p.1900).

The difficulties encountered by students without sufficient prior knowledge of the unit circle characterize challenges facing lecturers when assisting students to build firm background to be able to understand presentation on radian measurements (Mickey & McClelland, in press). This calls for the adaption and extension of lessons concerning the exploration of circle components. It is crucial to analyze whether such lessons might promote understanding of radian measurements prior the introduction of trigonometric concepts, ratios and functions. Thompson (2008) proposes a method by which mathematics educators can make implicit meaning clear and thereby address problems of instruction and curricula in a new light. He advocates for sequence and coherence in the curriculum.

THEORETICAL FRAMEWORK

In order for students to understand radian measurements they should be able to recognize a circle and its properties. The nature of radian is based on understanding properties of a circle such as circumference (C); radius (r) ; arc length $(\frown)$; central angle (θ); sectors and segments. Algebraic Insight Framework underpins this study. Learning mathematics entails understanding symbols and interpret them correctly to solve real-life and mathematical problems. Algebraic insight framework entails algebraic expectations and ability to link representations. In algebraic expectations, there is recognition of conventions, basic properties and identification of structure (Pierce & Stacey, 2001). In order for students to understand and master mathematics they need to know meaning of symbols; order of operations and properties of operations (Pierce & Stacey, 2001).

Recognition of conventions and basic properties

Learning of radian measurements depends much on understanding meaning of symbols used that represents the core concepts of a circle that are required in the learning of radian measurements. Key concepts that are required in the learning of radian measurements are: circle, circumference, diameter, radian, tangent, secant, chord, semi-circle, sector and segment.

The description of the components of a circle are provided below:

1. A circle is a set of all points equidistant from a fixed point.
2. Circumference is the distance around the edge of a circle.
3. Diameter is the measurement across the circle passing through the centre.

4. Radius can be a line from any point on the circumference to the centre of the circle.
5. Tangent is a line which touches a circle at just one point.
6. Secant is a line that intersects a circle at two points.
7. Chord of a circle is a straight line segment whose endpoints both lie on the circle.
8. Semicircle is half of a circle
9. Sector is the part of a circle enclosed by two radii of a circle
10. Segment of a circle is the region bounded by a chord and the arc subtended by the chord.

Identification of structure

In the learning of radian measurements students should be able to identify the nature of the given problem. They should be able to identify a circle, radii, arc length, and central angle. They should be able to deduce the formula that should be applied to bring a solution. In the figure 2 students should be able to use the diagram given to calculate the area of a sector, and area of the segment as the shaded region.

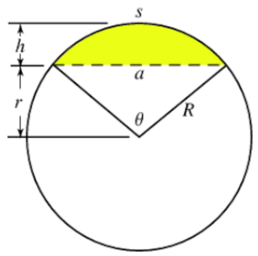

Figure 2: Sector: central angle; segment

Linking of symbolic; geometric diagram and numeric representation

Arcarvi's (1994) concept of 'symbol sense' can be used to describe the understanding of algebra required for working in partnership with technology (Pierce & Stacey, 2001). Symbol sense refers to students' ability to solve problems using mathematics, manipulative skills, and ability to formulate and interpret solutions (Pierce & Stacey, 2001). In the learning of radian measurements students should be able to link symbols, geometric diagrams and numerical representations. In figure 2, they should be able to see that the radius (r) of a circle is $6,$ the central angle is 60^0.

The main required understanding is that students should be able to deduce that from mathematics, a radian is defined by the formula: $\theta \equiv \frac{s}{r} rad.$ Students are expected to substitute the value of theta (θ) which is 60^0 and radius r which is 6 to calculate the value of arc length $s.$ Lastly students should be able to deduce a formula for finding area of a segment as shown below:

Area of a segment = Area of a sector minus Area of a triangle

The study shows how students should link their understanding of algebraic skills, circle geometry, trigonometry, and plane mensuration to study radian measurements. Plane mensuration deals with sides, perimeters and areas of plane figures of different shapes. The understanding of radian measurements requires students to be able to calculate areas of triangles, sectors, segments and able to substitute and make deductions when solving radian measurements.

RESEARCH DESIGN AND METHODS

The study engaged a case study design approach as an empirical inquiry that investigates an existing phenomenon in depth and within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident (Yin, 2009, p. 18).

The study employed qualitative research method to find out students' understanding in the learning of radian measurements. The study used interpretive paradigm as it is concerned with understanding the world as it is from subjective experiences of individuals. The study explores difficulties encountered by first year students registered for mathematics in an Electrical Engineering in the learning of radian measurements.

Research Participants

Participants of the study were thirty students registered for mathematics in their first year level in Electrical engineering. These were selected on the basis of being at risk students. At risk students are students who experience poor achievement in the schooling system (Siyepu, 2013, p.2). They are usually low academic achievers who show low confidence.

Data collection

Data was collected in the form of assessment tasks through administering three activities to be done in the classroom. These activities took about 90 minutes. In activity 1students explored the core concepts that should be understood to be proficient in radian measurements. In activity 2 students were instructed to convert angles from degrees to radians. In activities 3 students were assessed on mathematical word problems that need understanding of arcs, areas of sectors and segments involving understanding of radian measurements.

Activity 1: Give a description and illustration of the following concepts together with a symbol of each concept.

1. Circumference
2. Pi
3. Radians
4. Sectors

5. Arc length
6. Central angle
7. Radius.

Activity 2: Convert each of the angles from degrees to radians giving your answer to two decimal places.

2.1. 23^0
2.2. 77^0
2.3. 210^0
2.4. 315^0
2.5. 88^0

Activity 3: Calculate the unknowns

3.1 If $r = 15{,}8cm$ and $\theta = 23{,}20^0$; find s;

3.2. If $s = 30cm$ and $\theta = 135^0$, find r;

3.3 . If $r = 7.2cm$; and $s = 3cm$; find the central angle θ in degrees.

3.4. The centres of two circles with radii $4cm$ and $5cm$ are $6cm$ apart. Calculate the area of the region where the circles overlap.

3.5 . A lawn sprinkler located at the corner of a yard is set to rotate through 90^0 and project water out 30 metres. To three decimal places, what area of a lawn is watered by the sprinkler?

Data analysis

The first author marked the students' written work. The second author moderated the scripts to validate fairness and accuracy of students' procedures in their calculations. The scripts were sorted out and grouped together according to themes that emerge. Questions for clarity were based on the misconceptions done by students. Later students with common misconceptions were invited to share causes of their mistakes in calculations of radian measurements.

RESULTS OF THE STUDY AND DISCUSSION

Eight students were not able to give clear descriptions of the key concepts in the learning of radian measurements. Five students gave descriptions but they could not link descriptions with symbols. The fact that the area of a triangle is half base times height in a triangle: (Area of a $\Delta = \frac{1}{2}bh$). Students give the impression that they did not get clear explanation that the formula area of a $\Delta = \frac{1}{2}bh$ might not work in cases where the magnitudes of the base and height are unknown. In cases where the central angle and a radius are given students stuck to find the area of a triangle, area of a sector and area of a segment. This might be due to lack of schema level in the use of a sine rule to find the area of a triangle. In use of sine rule the area of any triangle $\frac{1}{2}ab\sin C$. This formula is useful if students do not know the height of a triangle (since you need to know the height for $\frac{1}{2}$ base $\times$ height). The sine rule is used when students are given either (a) two angles and one side, or (b) two sides and a non-included angle.

Three students who demonstrated understanding of procedures in calculations of converting angles from degrees to radians. These three students showed careless mistakes in their calculations where they could not cancel correctly despite that they demonstrated understanding of the correct formula to apply.

Twelve students did not know how to change degrees to radians. This suggests that lecturers should treat this section with care as it is taught for the first time among students registered for mathematics in their first year level in the field of Electrical engineering. Probably these students might have missed classes. At the same time students should be encouraged to work in groups to supplement their understanding with access to the more knowledgeable students.

Four students showed ignorance regarding the process of calculating area of a segment. They could not deduce that the area of a segment is denoted by area of a sector minus area of a triangle. This is aligned with exposure. These four students were not familiar with these kinds of mathematical problems.

Students' discussions demonstrated that they learn mathematics as isolated facts. They could not apply plane mensuration, trigonometry rules and circle geometry to solve a problem in radian measurements. The poor mathematical reasoning demonstrated by the students in the learning of radian measurements is aligned with the kind of teaching approaches applied in schools. The focus of teaching is rush to rules. As a result students showed an interest on how to tackle the problem. They showed no interest on getting meaningful understanding of concepts and their relationship to develop formulae.

The last two questions (3.4 and 3.5) were the most difficult problems to solve for the first year students studying radian measurements in their class of mathematics in the field of Electrical engineering. Seven students could not even attempt these questions. They could not translate words into mathematical statements. This might be associated with the background of students regarding the learning of radian measurements. At times students claim to have experience of not understanding word problems. These seven students claim that word problems were not thoroughly taught in their school levels.

The problems faced by students in problem solving can be located in two scenarios: (1) the language problem, many students who studied mathematics are not English first language speakers, as a result it becomes difficult for them to translate English from language statement to mathematical algebraic expressions. (2) Students enter university with inadequate mathematical vocabulary. As a result they struggle to make sense of mathematical problems, particularly, a situation of connecting circle geometry, plane mensuration, radian measurements and trigonometry.

IMPLICATIONS OF THE STUDY FOR TEACHING

The results of the study show that topics such as trigonometry, components of a circle and radian measurements in mathematics are taught as isolated ideas, as a results students were unable to make connections between core concepts of a circle and radian measurements. Students demonstrated understanding of the area of a triangle as half base times height. Despite that students were not aware why they were taught sine and cosine rules. The fact that they stuck when are required to calculate an area of a triangle without being given the length of the height and base. This suggests that teachers should emphasize that these rules are introduced to assist when a student has to calculate the area of a triangle with one angle given and other two sides that are not representing the height and base of a triangle.

The results of the study demonstrate that teachers should emphasize understanding of symbols that represent concepts in the learning of mathematics. Mostly mathematicians used symbols to demonstrate calculations. To many students these symbols do not make sense, as a result students attempt to master procedural aspects in calculations at the expense of conceptual development. There is a concern that in order for students to gain an understanding of mathematics there should be an emphasis on sense making in teaching and learning situation. The results of the study propose that students should be familiarized with calculations involving components of a circle as prerequisites for the understanding of radian measurements.

Despite that South African curriculum, Curriculum and Assessment Policy Statement (CAPS) document (2011) at matriculation level does not require students to be taught geometry up to rigour level of van Hiele's Theory, however, findings of this study suggest that students should be familiarized with applications of various topics to solve one kind of a mathematical problem. Students should be able to think about applying algebraic, geometric and trigonometric knowledge to solve a problem in a radian measurements. The study shows that students do not make sense of what they are learning in mathematics, as a result they did not reach a level of being able to apply the knowledge they have. Findings of the study echoes what Moore (2012) claims that in Georgia "students' and teachers' difficulties in trigonometry suggest that current curricular approaches to trigonometry do not engender coherent students understanding" (p.75). In a South African context there are areas such as calculation of the areas of a segment and sector that are not recommended to be taught in secondary schools. As a result students studying mathematics at the university first year level in the field of engineering struggle to grasp radian measurements as a prerequisite for calculus.

The findings of this study revealed that there is a need of developing students' understanding of mathematical concepts in relation with their symbols. Lecturers should be able to develop understanding of linking algebraic symbols with geometrical shapes, trigonometric ratios, and radian measurements. It is of utmost importance for lecturers to develop baseline assessment to be able to know the prior knowledge of students at entry level in a university. The crucial element of this study is that students at advanced levels such as first year university level should be able to connect geometric knowledge with trigonometric knowledge. The fact that students could not link or connect circle geometry with radian measurements and trigonometry suggests that students should be familiarized to mathematical problems that integrate various topics of mathematics to solve a given task. This study recommends thorough revision of measuring angles, analysis of circle geometry and its components, trigonometric ratios and trigonometric functions. Students should be familiarized with mathematical problems that lead to inter-connectedness of circle geometry, trigonometry and radian measurements to be able to obtain solutions of complicated radian measurements. There should be ample of word problems and problem solving activities incorporating the use of radian measurements to be taught in classroom discussions to familiarize students with translation of mathematical problems from words to mathematical symbols.

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ASCERTAINING AND UNDERSTANDING PRIMARY PRE-SERVICE TEACHERS' KNOWLEDGE OF TRANSFORMATION AND TESSELLATION IN GEOMETRY

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Profound or deep mathematical knowledge for teaching (MKT) – includes both content and teaching strategies – it means that the teacher possesses the required skills to engage the learner on different levels of abstraction, and has the ability to provide adequate mathematical explanation for what is observed in terms of learners' calculations or oral explanations. Literature suggests a close link between learner under-performance and the teacher's limited conceptual understanding of mathematics. Döhrman, Kaiser and Blömeke (2018: 57) highlight the fact that recent literature tends to put much more emphasis on the "knowledge base of mathematics teachers' classroom practice". This is especially relevant in terms of the "need to enhance the mathematical knowledge of students" expressed by Southwell and Penglase (2005:209) thirteen years earlier. This creates the impression that not much has changed over these past thirteen years in increasing students' mathematical knowledge for teaching. Mathematical knowledge for teaching (MKT) is considered essential to ensure success in terms of achieving set learning and teaching outcomes (Ball, Thames and Phelps, 2008, McAuliffe, 2013, Ndlovu, Amin and Samuel, 2017). The main aim of the recently initiated Primary Teacher Education Project in South Africa, of which the authors are active participants, is aimed at developing mathematics Teaching Units and Toolkits for training pre-service teachers in an effort to ensure quality teaching. It is within this context that this qualitative research takes place. The problem statement centres on ascertaining participating PSTs' (pre-service student teachers) knowledge and past experiences of particular mathematics content, in this case, transformation and tessellation as part of Geometry. The research question and sub-question that ultimately emerged were: What previous knowledge of transformations and tessellations do pre-service students possess? What experiences if any do pre-service students recall when transformations and tessellations were discussed or interacted with previously? The conceptual framework used for ascertaining participant pre-teachers' knowledge of transformations and tessellations is done in terms of what 'mathematical knowledge' they possess. This means the focus is on what knowledge of mathematical facts, what concepts, what procedures and what knowledge they have of the underlying relationships that exist. Furthermore, whether PSTs understanding of particular concepts are adequate, and whether related procedures are performed accurately, and what their understanding is of the conceptual foundations of that knowledge. A convenient sample of 50 second year registered Primary B.Ed. students at a University of Technology in the Western Cape participated in this study. Participants completed a questionnaire consisting of questions that reveal The questions in the self-developed and validated questionnaire consisted of a mixture of factual, closed- and open-ended questions. Participants' opinions and viewpoints were also sought. The questions were kept short and simple and an effort was made to only address one particular concept or idea in each to avoid ambiguity and confusion. Pre-service Teachers (PSTs) experiences, knowledge and viewpoints related to transformation and tessellation. The questionnaire was administered prior to the teaching of the content. Selective face-to-face interviews to find clarity on issues arising from the information in the questionnaires were conducted afterwards. Coding of data was used to review participants' responses to discover common themes and particular patterns in terms of similar and different experiences. Key words or concepts were identified and recorded to develop a sense of participants' background knowledge of transformations and tessellations. This case study research is considered relevant and crucial since the prime focus is on enhancing PST's understanding and knowledge of transformation and tessellation as it relates to the Curriculum and Assessment Policy Statement (CAPS). Findings revealed that the background knowledge as it related to transformation and tessellation to be very limited to totally absent in some, and the misconceptions that exist. The responses to the research questions to a large extent reveal PSTs' experiences, knowledge and viewpoints related to the concepts transformations and tessellations. There may be many valid reasons for the fact that the overwhelming majority of these participating students have no recollection of, or knowledge of what was supposed to be taught according to CAPS. This is cause for concern. Some of the reasons for this state of affairs could be that this content was never taught before; another, that it is too far back in the past to recall what specific content was taught, that the topic was taught in a way that did not actually arouse their interest as learners, nor made a lasting impression, and thus was easily forgotten. Chances are that these PSTs may not have encountered these two topics in a realistic (Freudenthal, 1991, Gravemeijer, 1994) manner, or in relation to other familiar contexts and concepts (Skemp, 1978), and lasting connections and associations might not have been made to enhance memory and remembering (Skemp, 1978). We also have to consider the possibility that the teaching approach to these topics might have been overwhelmingly instrumental as opposed to relational (Wubbles, Korthagen and Broekman, 1997). At this stage of their training (that is the second year of a four year program) these PSTs clearly have little to no CK and MKT (Ball, 2003) with respect to transformation and tessellation. For all intents and purposes these knowledge types are very limited or even seem to be non-existent. In this regard Venkat and Spaull's (2007) concern about teacher disposition and superficial or limited mathematics content knowledge becomes a really worrying aspect. Similarly the concept gaps and misconceptions of these PSTs that Ndlovu, Amin and Samuel (2017) referred to are real issues that trainers of these PSTs will have to contend with. The question is how will teacher trainers be able to enhance MCK and MKT (Ball, et al, 2008) within a short span of two years with only two lectures of 45 minutes each per week per year.These findings are to be used to inform the development of suitable training material for mathematics teachers. Literature reveals that PSTs' content knowledge is a critical focus for appropriately training or preparing mathematics teachers.

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DELIBERATING ABOUT ARTS-INTEGRATED RESOURCES TO DIMINISH DISCORDS IN MATHEMATICS

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INTRODUCTION

Mathematics performance amongst teachers and their learners in South Africa are far below comparable countries in the world (Venkat & Spaull, 2015). The responsibility to increase knowledge of pre-service students in mathematics should be shared by universities and schools so that the learning divide (Gravett, 2015) between theory and practice can be overcome. As intervention, the DHET PrimTEd project is presently developing comprehensive kits to be used by pre-service teachers during their work-integrated learning (WIL) to bypass the challenge of incompetent mentors and lack of suitable resources. Since the author has had considerable success with using arts-integration in the classroom, the possibilities of using it in a Mathematics kit is explored (Jansen van Vuuren, 2018).

The research question then asks; what are the prerequisites for a successful Mathematics kit for pre-service teachers? As sub-question; how can arts-integration be used to optimize outcomes? Accordingly, two main aspects covered in the literature search were potential impacting factors when designing a pre-service Mathematics kit and how arts-integration can augment Mathematics teaching.

Conspicuous factors which contribute to weak performances and could impact kit design were found to be language barriers (Azabdaftari & Mozaheb, 2012), the lack of pedagogical (PCK) and mathematical content knowledge (MCK) (Prestage & Perks, 2001) along with socio-economic (Fleisch, 2008) and emotive issues (Ashcraft & Moore 2009). The arts, which is often overlooked in Mathematics teaching, provides a solution to some of the contributors to surface learning (LaJevic, 2013).
Although much has been said in the literature about Mathematics teaching and learning, very little is available about kit design for pre-service teachers. Integrated-arts teaching is not new and is well-notated in the literature but not enough has been said about suitable methods of integration to ensure validity of both the subject being taught (Mathematics) as well as the art form being integrated.

OBJECTIVES

The objectives with this study are to explore:

- factors influencing Mathematics teaching and learning
- considerations when designing a contextually relevant kit for pre-service teachers
- how arts-integration should be used to augment Mathematics teaching
- the design of a suitable arts-integrated lesson plan and resource kit for pre-service teachers.

METHODOLOGY

The study uses qualitative research and a pragmatic paradigm to explore literature for guidelines and innovative thinking surrounding mathematical kits for pre-service teachers. Knowledge gained from the literature is augmented with observation of pre-service education students during WIL practice over a five year period. Pragmatism is seen through the eyes of Goldkuhl (2012:2) who says [it is] 'concerned with action and change and the interplay between knowledge and action.' This view of pragmatism aligns with the aim of the study since it reports on the knowledge gained through the literature and how it was actioned to compile resources for pre-service educators to bring about change.

The work group started with intensive PrimTEd workshops which acted as incubation hubs and inspired innovative thinking and perspectives. Literature search about topics along with their misconceptions and errors played an important role to gather relevant information. Members also worked independently and shared efforts on Google Drive for peer review. A kit format was not initially prescribed and styles are still changing and being adapted as the process evolves.

RESULTS

There is a strong possibility that a well-designed kit along with arts-integrated teaching can be beneficial to alleviate most of the impacting factors causing low achievement in Mathematics in the primary school. The actual kit design is evolving into a product that could make a change in Mathematics acquisition when pre-service teachers are doing their WIL. Final claims are not possible since this is an ongoing project that will eventually be tested before being put into use. Examples of the type of work that has been emanating from this project shows some of the possibilities of integrating arts into a Mathematics toolkit.

CONCLUSION

Producing a suitable Mathematics package in the South African context, requires deeper thinking than just the obvious content to ensure that the kit will not become a shelf decorator. Important considerations for the kit include;

- suitable teaching methods (arts-integrated) to lessen the fear for Mathematics
- a suitable level of language that is not marred with jargon,
- detailed MCK and PCK (including integrated approaches to show each step and level required to teach the topic successfully) without providing too much reading for the students,
- enough portable and economic resources to encourage innovative teaching
- flashcards covering all the important vocabulary.

When all the above elements are contained in a sturdy and well-presented mathematics kit, many variables will be eliminated which cause varying standards and incorrect teaching methods and better outcomes can be expected where pre-service educators are doing WIL. This same kit could be used to train in-service teachers.

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LECTURERS' CHALLENGES IN THE IMPLEMENTATION OF E-LEARNING PRACTICES IN TEACHING AT A FACULTY LEVEL IN HIGHER EDUCATION

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Use of e-Learning practices for teaching purposes has considerably increased in the past three decades; hence, institutions of higher learning have developed interest in transforming their teaching practices using technology. The e-Learning practices nowadays, plays fundamental role of the instructional delivery method intended at students' empowerment and learning facilitation process. However, university lecturers require thorough pedagogical expertise, and necessary skills to make a real difference in enhancing their students learning. Lecturers have to learn appropriate pedagogical skills to be able to integrate information technology in a comprehensive approach into educational practice (Voogt, Knezek, Arvaja, Häkkinen, & Kankaanranta, 2008).

This paper is an investigation on implementation challenges of e-Learning practices faced by lecturers in the University of Technology (UoT) sector. The study objective is to make a meaningful contribution to the integration of technology in the facilitation of teaching in developing countries like South Africa. We believe that experience in the integration of technology for teaching, is critical to enable lectures' transition from traditional delivery methods of chalk and duster to modern education level, which relies on e-Learning practices. Overall, the study is governed by the following research questions: (1) to what extent do lecturers use e-Learning resources in their teaching practices? and (2) what are the e-Learning implementation challenges faced by lecturers in the University of Technology and how these can be addressed?

Several researchers have defined e-Learning in many different ways, for this study we follow definition: "e-Learning as instruction delivered on a digital device such as a computer or mobile that is intended to support learning" (Clark & Mayer, 2003). E-Learning serves as a major tool that involves innovative technologies through digital and online media, with the learning environment accessible to students anywhere and anytime, to attain knowledge at their own pace. E-learning as a practice of application of information technology in teaching and learning has been adopted by many higher education institutions and become an essential part in delivering a learning experience (Sadik, 2007).

Much of the research has focused on e-Learning practices as the platform that encourages student learning and ultimately increase their level of engagement (Hiltz, 1993); (Wang & Wang, 2009); (Wang, 2010); (Hardaker & Singh, 2011); (Macharia & Pelser, 2012). based on the literature, very little research has been undertaken on lecturers' challenges in the implementation of e-Learning practices in teaching at a faculty level in higher education (Steel & Hudson, 2001); (Eynon, 2005) ; (Olaniran, 2006).

The relatively scarce research on the lecturers' challenges in the implementation of e-Learning practices has left opportunity for further research. In fact, there is a need to conduct further research on how UoTs deal with implementation challenges of e-Learning practices in the facilitation of teaching and learning and also address issues of influencing lecturers' commitment to the integration of technology, and how this commitment should be achieved by institution management (Macharia & Pelser, 2012), (Macharia & Nyakwende, 2010). However, the current e-Learning practices in the higher education institutions in particular in developing countries are not fully enhanced.

In the research conducted by (Andersson & Grönlund, 2009) which was based on the critical challenges of e-Learning implementation with the primary focus on developing countries, they identified four categories of specific challenges, namely courses, individuals, technology and context. Their research concluded that the challenges are equally valid for both developed and developing countries although in developing countries, the issue was with access to technology and context, where as in developed countries, challenges point to individuals. A study by Omidinia, Masrom, & Selamat (2011) in Iran on e-Learning infrastructure in developing countries found that e-Learning success depends on implementation strategies, technology focus, an open source technology and on one-time funding.

The aim of introducing the concept of integration of technology in higher education institutions is to enable lecturers to engage with diverse teaching challenges through e-Learning practices. However, lecturers intending to integrate technology in the institutions, experience various implementation challenges such as learning how a specific new tool works, how they can integrate the associated teaching materials didactically. In these instances, lecturers would need to have a certain degree of competence in the technology-associated tools.

Lecturers not used to technology in their teaching practices, or have not personally used it, will find it difficult to draw from their previous experiences or from their recent introduction to the technology platform. In order to use learning technologies effectively, lecturers need to experience its affordances to teaching process. Continuous professional development is important to familiarize them to new technologies, make sure of persuasive adoption, and afford experience with the technology. However, effective lecturers' development interventions ought not to focus on the mechanistic and technical viewpoints of e-Learning only, it must put emphasis on applicable pedagogy, concentrate on particular teaching beliefs, afford real life interactions as their online students would, and contextualize the professional development to address their needs.

Internal structures and contexts of the higher education institution should allow lecturers to try-out and adopt new pedagogical methods in order for their attitude to change towards integration of technology. University management should accommodate integration of technology across the institution in their strategic planning, they must provide facilities for lecturers to develop interest into teaching and learning using technology platform and provide assistance throughout all administrative, technical, and pedagogical areas (Elton, 1999; Riel & Becker, 2008).

The participants of the study were 27 lecturers that do not have any experience in using technology in the classroom, 6 lecturers that have reasonable experience in e-Learning practices at the university. Purposive sampling was considered to choose respondents in various categories on the implementation of e-Learning in teaching in the faculty. A total of 35 questionnaires were distributed in a batch of five to lecturers with the assistant of the Heads of Department in each of the 6 departments identified from the faculty.

The data was collected by administering a first structured questionnaire with open-ended questions within 4 binary and 5 Likert scale data to the 35 lecturers that do not have an experience in teaching using technology Finlayson et al. (2006). The survey focused on the faculty of Natural Sciences academic staff members in order to establish possible differences in the use of e-Learning. This first questionnaire covered the following themes: Frequency of use of e-Learning by lecturers, lecturers' satisfaction with access to e-Learning resources, satisfaction with the support available to assist use of e-learning, lecturers' views on current experience of using e-Learning and lecturers' confidence in using e-learning

The study findings indicate that lecturers display varying degrees of enthusiasm towards the integration of technology into education; however, they encountered many implementation challenges. The key challenges were lack of confidence, lack of access to resources and poor infrastructure in the teaching environment. Lecturers are under increasing pressure from their students to present flexible technology enhanced programme delivery but they themselves are struggling with technology integration. If this University would like to progress systematically in improving the range and quality of teaching and learning, it has to define logical strategic e-learning frameworks which involve the establishment of satisfactory support components and processes that promote the development of technology related competences.

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LEARNING GEOMETRY ONLINE: A CREATIVE INDIVIDUAL LEARNING EXPERIENCE

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This paper is part of a larger study that looked into the experiences of student teachers when using a form of online technology, which utilised meaningful interactions to teach high school geometry. The study was conducted with ten mathematics student teachers who were studying at a South African university, with the aim of examining student teachers' perceptions of learning geometry online. This case study sought to capture the students' knowledge of technologies when teaching circle geometry through a careful implementation of Geometric Habits of Mind and an Instructional Design model. The intention of the support programme was that students will reproduce their learning experience, during this study, in their classrooms. This paper reports on the design of learning circle geometry proofs online and key findings on the perceptions of technology to Geometry: Knowing the essentials, A tool in learning geometry and A catalyst to learning geometry.

USING TECHNOLOGY FOR TEACHING AND LEARNING STRAIGHT LINE GRAPHS

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There is concern regarding mathematics results and the pedagogy used within the classroom. Society has seen swift development with a call for these changes to be reflected within the classroom. New methods of teaching mathematics are being sought with the purpose to improve teaching and learning while making mathematics relatable to the new generation of students (Birgin, 2012, p. 140). The incorporation of technology within the classroom has been seen as the possibility to accomplish this change. The purpose of this study was to determine how effective the GeoGebra app is in allowing students to successfully discover the properties of straight line graphs and student's responses to the use of the app. A mixed method design was used in this study with data generated through an investigation, task, and individual interviews. Results of the study show that GeoGebra aided students successfully in discovering the properties of straight line graphs with the large majority of students understanding both concepts. Further to this, the results show that students had a positive outlook to the use of the app and enjoyed the experience. Likewise, they feel that the app helped them to better understand the concept.

AN ETHNOMETHODOLOGICAL ANALYSIS OF THE STUDENTS' UNDERSTANDING OF THE CONCEPT OF TRIGONOMETRY IN HIGH-STAKES EXAMINATION IN SOUTH AFRICA

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The purpose of this study was to investigate and analyze the responses of examinees in the NSC mathematics examinations meaningfully in the concept of trigonometry. While the study used an ethno-methodological approach in the main, a documentary analysis approach was also adopted. Documentary analysis was necessitated by the private nature of the NSC examination to the extent that we had access only to the written work of the examinees. The major findings were: (1) that the strategies and tactics used by examinees are highly driven by the context within which the high-stakes examination is situated. (2) that examinees' ways of working exhibit the general structure of the practice that is commonly found in mathematic discourse practices. Further studies are required to deepen the understanding of the thinking processes of examinees through conducting focus group interviews where the examinees are afforded opportunities to explain their workings.

AN EXPLORATION OF THE USE OF TECHNOLOGY IN TEACHING MATHEMATICS IN A GRADE 10 CLASS: A SELF-STUDY

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This study sought to explore the use of technology in the teaching of mathematics in a grade 10 class. Over the past few years the literature has shown that the appropriate use of technology when teaching mathematics has the potential for improving learners' performance. In the light of the widespread poor performance of learners in mathematics, this study particularly focused on the teaching of grade 11 Euclidean Geometry as it is the biggest section in mathematics with which learners struggle. Evidence from various research studies as well as personal experience regarding the poor learner performance in Euclidean Geometry has raised concern. Thus, this study came about as an attempt to respond to the following research question: How do grade 10 learners respond to the learning of grade 11 Euclidean Geometry using technology?